IL
NUOVO CIMENTO
ORGANO DELLA SOCIETÀ ITALIANA DI FISICA
SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOL. XVIII, N. 4 *Serie decima* 16 Novembre 1960

On Certain Representations of the Dirac Equation.

A. CHAKRABARTI

National Physical Laboratory - New Delhi

(ricevuto il 2 Maggio 1960)

Summary. — By a generalization of the transformation given by Eriksen some representations of the Dirac equation are derived, particularly cases including the well known ones of Foldy and Wouthuysen and of Cini and Touschek. A modification of the procedure gives a transformed Hamiltonian $\alpha_i E$ $(i=1, 2, 3)$, such that Γ_5 as defined by Cini-Touschek, has exactly the same form (namely $-\sigma_p\lambda$), as in their case. Some special features of this representation are discussed.

1. – Derivations.

The main purpose of this paper is to derive and examine a representation in which the Hamiltonian commutes with γ_5, $\Gamma_5\,(\equiv U^*\gamma_5 U)$ has the same significance as in the case of CINI-TOUSCHEK [1], and yet the relativistic connection between the velocity and momenta is preserved.

(*A*): Since the procedure is based on the paper of ERIKSEN his procedure will be briefly indicated, and at the same time generalized.

Let ζ be an hermitian operator, such that

$$\zeta^2 = 1\,. \tag{1.1}$$

Let U be the unitary operator effecting the transformation,

$$\Psi' = U\Psi\,, \qquad H' = UHU^*\,. \tag{1.2}$$

(1) M. CINI and B. TOUSCHEK: *Nuovo Cimento*, **7**, 422 (1958).

We want that,

$$[\zeta, UHU^*] = 0 \qquad \text{or,} \qquad [U^*\zeta U, H] = 0\,. \tag{1.3}$$

If in Eriksen's method([2]) we replace β by the more general operator ζ, we can derive exactly analogously, the solution (under the same assumptions and restrictions),

$$U = \tfrac{1}{2}(1 + \zeta\lambda)[1 + \tfrac{1}{4}(\zeta\lambda + \lambda\zeta - 2)]^{-\frac{1}{2}}\,, \tag{1.4}$$

$$U^* = \tfrac{1}{2}(1 + \lambda\zeta)[1 + \tfrac{1}{4}(\zeta\lambda + \lambda\zeta - 2)]^{-\frac{1}{2}}\,. \tag{1.5}$$

Other corresponding formulas can be derived similarly and in addition, one can show that if

$$[\zeta, H^2] = 0 \tag{1.6}$$

then

$$H' = UHU^* = \zeta(H^2)^{\frac{1}{2}}\,. \tag{1.7}$$

In the free particle case, when

$$H = \beta m + \boldsymbol{\alpha}\cdot\boldsymbol{p}\,. \tag{1.8}$$

Putting $\zeta = \beta$, $(\boldsymbol{\alpha}\cdot\boldsymbol{p})/p$ respectively, we get the F-W ([3]) and C-T ([1]) representations. Evidently we can put many other operators for ζ such as γ_5, σ_i etc., but we briefly mention only the case $\zeta = \alpha_i\,(i = 1, 2, 3)$. In this case $H' = \alpha_i E$ $(E \equiv \sqrt{m^2 + p^2})$, which is the same as the Hamiltonian we will derive finally by another method. But here $\Gamma_5\,(\equiv U^*\gamma_5 U)$ does not have any simple significance and hence we do not discuss the above representations. Indeed

$$\left.\begin{aligned} &U = \exp\,[i\gamma_i(\theta'/2)]\cdot\exp\left[i\,\frac{\gamma_j p_j + \gamma_k p_k}{(p_j^2 + p_k^2)^{\frac{1}{2}}}\,(\theta/2)\right], \\ &\theta = \mathrm{tg}^{-1}\,\frac{(p_j^2 + p_k^2)^{\frac{1}{2}}}{m}\,, \qquad \theta' = -\,\mathrm{tg}^{-1}\,\frac{(m^2 + p_j^2 + p_k^2)^{\frac{1}{2}}}{p_i}\,, \\ &\qquad\qquad (i, j, k = 1, 2, 3, \text{ and no summation is implied}) \end{aligned}\right\} \text{with} \tag{1.9}$$

and also other expressions give the same H' all of which are discarded for the same drawback.

([2]) E. Ericksen: *Phys. Rev.*, **111**, 1011 (1958).

([3]) L. L. Foldy and S. A. Wouthuysen: *Phys. Rev.*, **78**, 29 (1950).

(*B*): So long as we do not assume any particular form of H, it seems that we can only start by postulating like ERIKSEN,

$$U^*\zeta U = \lambda\left(\equiv H/(H^2)^{\frac{1}{2}}\right) \tag{1.10}$$

so that the condition $[U^*\zeta U, H] = 0$ is automatically fulfilled. But as soon as we choose a definite expression for H, it may be possible to find other Hermitian operators, with square unity, which commute with H.

Thus for $H = \beta m + \boldsymbol{\alpha}\cdot\boldsymbol{p}$ we can choose

$$U^*\zeta U = \sigma_p\left(\equiv \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p}\right). \tag{1.11}$$

In this case, proceeding exactly as before, we have

$$\begin{cases} U &= \frac{1}{2}(1 + \zeta\sigma_p)[1 + \frac{1}{4}(\zeta\sigma_p + \sigma_p\zeta - 2)]^{-\frac{1}{2}}, \\ U^* &= \frac{1}{2}(1 + \sigma_p\zeta)[1 + \frac{1}{4}(\zeta\sigma_p + \sigma_p\zeta - 2)]^{-\frac{1}{2}} = \sigma_p\zeta U\,. \end{cases} \tag{1.12}$$

All the features being similar, we proceed at once to a particular case.

Let

$$\zeta = \sigma_3\,. \tag{1.13}$$

Then

$$\begin{cases} U &= \dfrac{1}{\sqrt{2}}(1 + \sigma_3\sigma_p)(1 + p_3/p)^{-\frac{1}{2}} \\ U^* &= \dfrac{1}{\sqrt{2}}(1 + \sigma_p\sigma_3)(1 + p_3/p)^{-\frac{1}{2}} \end{cases} \tag{1.14}$$

and

$$H' = UHU^* = (\beta m + \alpha_3 p)\,, \tag{1.15}$$

where, we note that

$$U^*\gamma_5 U = \gamma_5 \tag{1.16}$$

and

$$U^*\sigma_3 U = \sigma_p\,. \tag{1.17}$$

Now to H' let us apply a C-T type transformation.

Let

$$V \equiv e^{is'} = (\cos\theta'/2 + \beta\alpha_3 \sin\theta'/2) \quad \text{where} \quad \cos\theta' = p/E\,, \quad \sin\theta' = -m/E. \tag{1.18}$$

Then

$$H'' = e^{is'} H' e^{-is'} = \alpha_3 E \,. \tag{1.19}$$

Also

$$\Gamma_5' = e^{-is'} \gamma_5 e^{is'} = -\sigma_3 \lambda' \qquad (\lambda' \equiv H'/E), \tag{1.20}$$

$$\Gamma_5 = U^* \Gamma_5' U = -(U^* \sigma_3 U)(U^* \lambda' U) = -\sigma_p \lambda \qquad \text{(from (1.17))}, \tag{1.21}$$

which is exactly the same as the Γ_5 of C-T.

The same steps can be gone through with $\zeta = \sigma_1, \sigma_2$ as well, and so we have, for the transformed representation,

$$\left\{ \begin{array}{ll} & H_{\text{tr.}} = \alpha_i E \qquad (i = 1, 2, 3) \\ \text{so that} & \\ & [H_{\text{tr.}}, \gamma_5] = 0 \quad \text{and} \quad \Gamma_5 = -\sigma_p \cdot \lambda \,. \end{array} \right. \tag{1.22}$$

2. – Discussions.

As already mentioned, the representation finally derived has the two essential features of C-T representation (namely $[H_{\text{tr.}}, \gamma_5] = 0$ and $\Gamma_5 = -\sigma_p \cdot \lambda$. The improvement consists in that we have in this case,

$$\dot{\boldsymbol{x}} = i[H_{\text{tr.}}, \boldsymbol{x}] = \alpha_i \boldsymbol{p}/E \qquad (i = 1 \text{ or } 2 \text{ or } 3). \tag{2.1}$$

Thus the eigenvalues of $\dot{\boldsymbol{x}}$ are $\pm \boldsymbol{p}/E$ according as the energy is $\pm E$, since $H_{\text{tr.}} = \alpha_i E$.

In this case of the C-T representation,

$$\dot{\boldsymbol{x}} = i[\alpha_p E, \boldsymbol{x}] = \alpha_p \cdot \frac{\boldsymbol{p}}{E} + \frac{E}{p^3} \boldsymbol{p} \times (\boldsymbol{\alpha} \times \boldsymbol{p}) \,. \tag{2.2}$$

The first term on the right hand side has eigenvalues of the order p/E which is correct, but for the second one they are of the order of $E/p > 1$ (for $m \neq 0$), *i.e.* greater than the speed of light. To make the situation clear we consider a simple case, though the feature is perfectly general and does not depend on it. If we consider a simultaneous eigenfunction of the mutually commuting set of operators, $\dot{x}_1$, p_1, p_2, p_3 (though such eigenfunctions may be difficult to obtain in practice) and consider the special case in which belongs to the eigenvalues zero of p_1 and p_2 and p of p_3 say, the eigenvalues of $\dot{x}_1$ are

the same as those of the operator $(E/p)\alpha_1$, *i.e.* they are $\pm E/p$. Though in this region $E/p \to 1$, for any fixed value of p we can increase m and exceed more and more the velocity of light. With $\dot{x}_2$ also we meet this same paradoxical features. This is specially undesirable in a representation proposed specifically for the relativistic region. It is to be noted that preserving the usual relativistic expression for velocity by introducing projection operators, as has been done by BOSE *et al.* [4] is to prove no special advantage of the representation since the same may be done in the usual Dirac representation. Moreover we lose then the commutativity of the components of $\boldsymbol{x}$ which is undesirable. Indeed relations analogous to most of those given in the table of [4] can be derived in the usual representation itself. Those features are hence characteristic, not of the representation, but of the use of projection operators.

Considering the operators $\boldsymbol{X} \equiv U^* V^* \boldsymbol{x} V U$, of which the full expressions can be derived as usually, we note the following features. The operators $(X_i - x_i)$ commute with $\boldsymbol{p}$. In the special case when $m = 0$ and $p_1\psi = 0 = p_2\psi$ the eigenvalues of $(X_3 - x_3)$ are zero, those of $(X_2 - x_2)$ and $(X_1 - x_1)$ are the same as those of $\frac{1}{2}\sigma_1/p$ and $-\frac{1}{2}\sigma_2/p$ respectively. This can probably be related to the fact that for zero rest mass the longitudinal component of the zitterbewegungen vanishes. We can make a rough semi-classical picture in which the zitterbewegung is confined to the transverse plane, so that in order to get $\boldsymbol{p}/E$ as velocity we have to modify the eigenvalues of the transverse but not those of the longitudinal components. If we think in terms of some model connecting spin with zitterbewegung, *e.g.* that proposed by KOBA [5], the above feature can be taken to provide a rough and intuitive explanation of the fact that for zero rest mass the spin is always parallel or antiparallel to the momentum. This feature too cannot be brought out in C-T representation since, in that case, for $m = 0$, U reduces to the identity transformation.

The last two features discussed are present is the F-W representation too. But it should perhaps be mentioned here that the expressions for $\boldsymbol{X}$ or $\boldsymbol{x}'$ given in [3] differ from those given by PRYCE [6], by a factor p/E in the third term, though the contrary is specifically stated. Actual calculation shows that it is Pryce's result that is printed correctly.

We can conclude by saying that the representation proposed has, along with the essential features of the C-T representation, additional properties which render it more suitable for the relativistic region. Of course it does not solve the difficulties inherent in such representations, involving non-locality

---

[4] S. K. BOSE, A. GAMBA and E. C. G. SUDARSHAN: *Phys. Rev.*, **113**, 1661 (1959).
[5] Z. KOBA: *Progr. Theor. Phys.*, **4**, 319 (1949).
[6] M. H. L. PRYCE: *Proc. Roy. Soc.*, A **195**, 62 (1949).

of the operator and other features, which have been pointed out by CINI and TOUSCHEK at the conclusion of their paper.

* * *

I am grateful to Professor K. S. KRISHNAN, F.R.S., Director, National Physical Laboratory, for the interest he has shown in the work and for the encouragement given.

RIASSUNTO (*)

Con una generalizzazione della trasformazione esposta da Eriksen si derivano alcune rappresentazioni della equazione di Dirac; i casi particolari comprendono quelli ben noti di Foldy e Wouthuysen e di Cini e Touschek. Una modifica del procedimento dà un Hamiltoniano trasformato $\beta_i E$ ($i=1, 2, 3$), tale che Γ_5, quale definito da Cini-Touschek, ha esattamente la stessa forma (cioè $-\sigma_p \lambda$) che nel loro caso. Si discutono alcune caratteristiche speciali di questa rappresentazione.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVIII, N. 4 16 Novembre 1960

Central and Peripheric Interactions of 7.3 GeV Negative Pions in Emulsions.

E. M. Friedländer, M. Marcu and M. Spîrchez

Cosmic Ray Laboratory, Institute of Atomic Physics - Bucharest

(ricevuto il 27 Maggio 1960)

Summary. — 534 interactions of 7.3 GeV negative pions in an emulsion stack have been studied. Average multiplicities of heavy prongs and shower particles were obtained. Individual Lorentz factors γ_c were computed for 395 showers with $n_s \geqslant 3$. A detailed study is made of the γ_c-spectrum of events from light nuclei. This spectrum appears structured into head-on collisions with one or two nucleons and peripheric collisions involving one or two pions in the struck nucleon's meson cloud. Pion-pion collisions show a marked « two-centre » structure, attributed tentatively to the formation and isotropic two pion decay of pionic resonant (isobaric) states of mass ~ 3 μ. The e^--e^+-pair spectrum from π^0-decay has been measured and yields an average inelasticity of $\sim 75\%$ in π^--nucleus collisions.

---

1. – In a previous paper of one of us [1] it has been shown that, at a given primary energy γ_0 the spectrum of values of the c.m. system Lorentz factor γ_c for a large number of meson showers produced by high-energy nuclear-active particles, can provide a means for probing the structure of the target particle. Meson production by 9 GeV protons was thus shown to proceed via a tunnel mechanism, and preliminary evidence was obtained for the occurence of interactions in which the effective target mass appeared to be only « part of a nucleon », *viz.* of the order of the pion rest mass. A similar conclusion was reached independently by Birger and Smorodin [2] from the analysis of dif-

---

(1) E. M. Friedländer: *Nuovo Cimento*, **14**, 796 (1959).

(2) N. Birger and Yu. Smorodin: *Žurn. Eksp. Teor. Fiz.*, **37**, 1355 (1959).

ferent accelerator- and cosmic ray results, by studying the lab. system energy and angular distributions of the secondary particles.

We report hereafter the results of an emulsion experiment with high-energy pions, which, owing to their short de-Broglie wave-length and their relative « elementarity » (as compared to protons) would be expected to be even better fitted for probing the nucleon's structure.

2. – A stack of 100 NIKFI-R pellicles, $(10\times20\times0.04)$ cm^3 each, was irradiated in the 7.3 GeV π^--beam of the JINR proton synchrotron at Dubna, to a total dose of $1.3\cdot10^4$ pions/cm^2. To date 534 interactions of the beam pions where detected by area scanning under $10\times1.5\times20$ magnification. The multiplicity distribution of charged shower particles ($g \leqslant 1.5\, g_0$) is shown in Fig. 1 The average multiplicity is

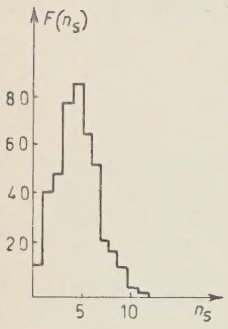


Fig. 1.

$$\overline{n}_s = 4.1 \pm 0.1 \,. \tag{1}$$

The dependence of $\overline{n}_s$ on the number N_h of heavy prongs is shown in Fig. 2 and proves that no appreciable bias is introduced by the loss of $(N_h = 0)$-stars. The average multiplicity of heavy prongs is

$$\overline{N}_h = 10.7 \pm 0.6 \,, \tag{2}$$

wherefrom we deduce, using the empirical Bristol formula [3], the average energy transferred to the target nucleus

$$\overline{E}_h = (0.85 \pm 0.07)\ \text{GeV} \,. \tag{3}$$

On each star with at least three shower particles in the forward hemisphere (« shower »), the space angles θ of the shower particles with respect to the incident pion were measured. For each shower an estimate ξ for $\log \gamma_c$ was computed by means of the Castagnoli formula [4]

$$\xi \equiv \overline{\log \operatorname{ctg} \theta} \,. \tag{4}$$

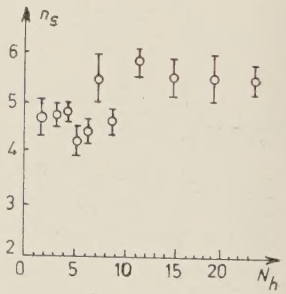


Fig. 2.

[3] R. BROWN, U. CAMERINI, P. FOWLER, N. HEITLER, D. KING and C. POWELL: *Phil. Mag.*, **40**, 307 (1949).

[4] C. CASTAGNOLI, G. CORTINI, C. FRANZINETTI, A. MANFREDINI and D. MORENO: *Nuovo Cimento*, **10**, 1539 (1953).

The ξ-distributions obtained are shown in Fig. 3, separately for heavy (H) nuclei ($N_h > 4$), and light (L) nuclei ($N_h \leqslant 4$), (cf. [1]) and also Fig. 2). The statistics for L-nuclei were increased by an additional selective scan for stars with $N_h \leqslant 4$ only. Thus, the ratio L/H of light to heavy nuclei events in Fig. 3 is not representative for a random sample. The true observed ratio was

$$\frac{L}{H} = \frac{55}{243}.$$

The arrows in Fig. 3 show the expectation value of ξ for different values of the ratio

$$\nu \equiv \frac{m_t}{\mu}, \tag{5}$$

where m_t is the assumed target mass, and μ the pion rest mass. These expectation values were corrected for the non-monoergetic c. m. spectrum of the secondaries according to ([5]) and ([1]) ([*]).

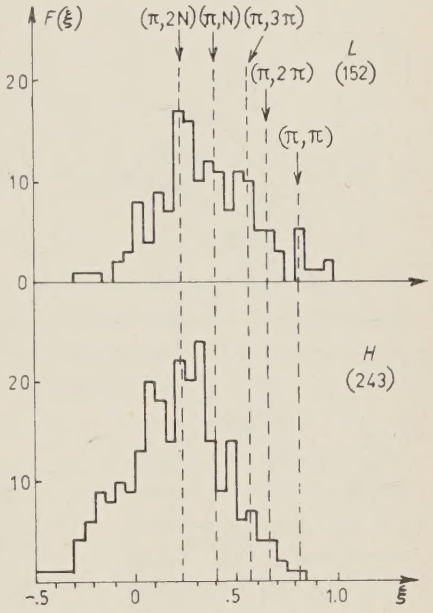


Fig. 3.

As expected, the H-distribution shows a marked shift towards higher ν-values, as compared to the L-distribution. The latter is visibly structured into central collisions of the incident pion mainly with tunnels containing one or two nucleons, and peripheric interactions with apparent target masses of $3\,\mu$ and $1\,\mu$. Since peripheric collisions are expected to yield relatively low multiplicities, the L-diagram was split up into events with $n_s < 5$ and $n_s \geqslant 5$, Fig. 4. As can be seen, the head-on collisions with one nucleon are almost absent in the $n_s < 5$ histogram, which contains a large proportion of events with m_t inferior to the nucleon mass M. The same conclusion is borne out by the ξ-distributions shown in Fig. 5 where the data have been split up into events with high inelasticity ($\sum \operatorname{cosec} \theta > 15$) and low inelasticity ($\sum \operatorname{cosec} \theta < 15$)

([5]) E. M. FRIEDLÄNDER: *Nuovo Cimento*, **12**, 483 (1959).

([*]) The validity of the correction formula derived in ([5]) has been proved by recent measurements ([6,7]).

([6]) P. JAIN, E. LOHRMANN and M. TEUCHER: *Phys. Rev.*, **115**, 643 (1959).

([7]) H. WINZELER, B. KLAIBER, W. KOCH, M. NIKOLIĆ and M. SCHNEEBERGER: preprint (1960).

according to [8]. The correct attribution of the (π-2$\mathcal{N}$) and (π-$\mathcal{N}$) peaks to the corresponding ν-values is confirmed by the angular distributions of the showers belonging to these peaks shown in Fig. 6 as (integral) Duller-Walker diagrams [9], which are clearly consistent with isotropy (straight lines of slope two).

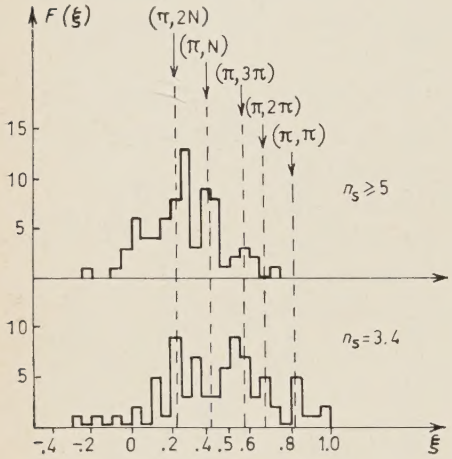


Fig. 4.

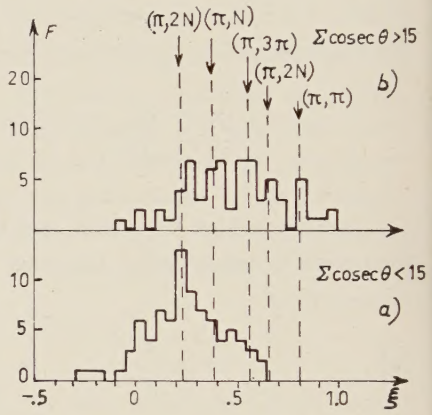


Fig. 5.

The statistical degree of significance for the reality of the peaks corresponding to $m_t < M$ has been derived as follows:

Assume that only (π-2$\mathcal{N}$) and (π-$\mathcal{N}$) collisions are present and the $m_t < M$ peaks are due to statistical fluctuations only. In this case the ξ-distribution ought to be a superposition of two gaussians with standard deviation

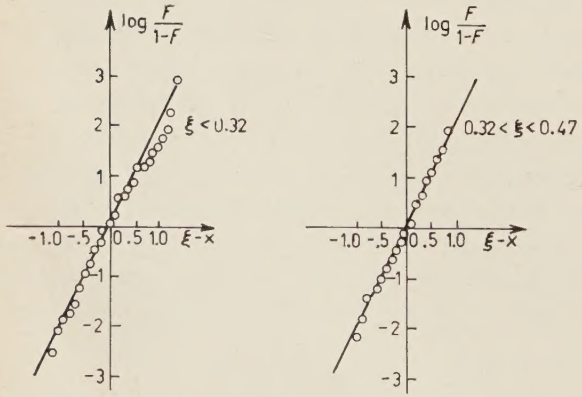


Fig. 6.

$$\sigma = \frac{0.36}{\sqrt{\bar{n}_s}}, \tag{6}$$

each [1]. Using the observed amplitudes of the (π-$\mathcal{N}$)

(8) E. Edwards, J. Losty, D. Perkins, K. Pinkau and J. Reynolds: *Phil. Mag.*, **3**, 237 (1958).

(9) N. Duller and W. Walker: *Phys. Rev.*, **93**, 215 (1954).

and (π-2$\mathcal{N}$) peaks (*), the combined standard deviation expected for the whole ξ-diagram turns out to be

$$\langle \sigma_t \rangle = 0.175 \,, \tag{7}$$

whereas the observed value

$$\sigma_{\text{obs}} = 0.247 \pm 0.014 \tag{8}$$

lies 4.6 standard errors away. Thus, events with $m_t < M$ are present in a statistically significant amount.

3. – Fig. 7-*a* shows the Duller-Walker plots for L-events with $\xi > 0.75$. This group, clearly distinct from the rest of low-multiplicity L-events should, formally, be attributed to « π-π » collisions in the fringe of the target nucleons meson cloud. The diagram shows the characteristic bend of the cosmic-ray two-centre jets ([11,12]). According to ([12]) it was therefore decomposed into forward and backward cones (Fig. 7-*b*). This diagram is clearly consistent with two centres of equal mass, moving away from each other in the c. m. system with a Lorentz factor

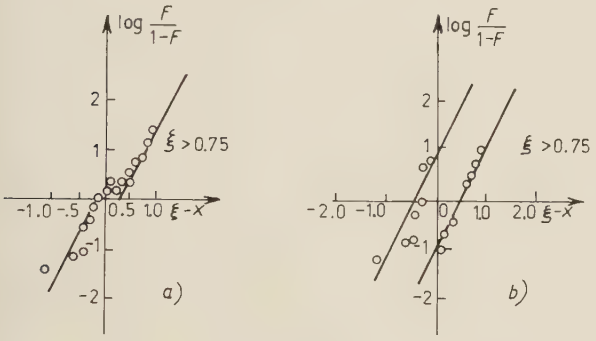


Fig. 7.

$$\bar{\gamma} = 1.9 \pm 0.2 \tag{9}$$

and isotropic decay in their own rest system. The mass m^* of these centres can now be computed from energy conservation, and turns out to be

$$m^* = (2.7 \pm 0.4)\ \mu \,. \tag{10}$$

(*) Comparison of these amplitudes yields an interaction cross-section in nuclear matter for the incident pions ~ 0.83 times the geometrical one. This corresponds to an interaction mean free path of 32.5 cm of emulsion, in good agreement with previous results ([10]) under similar conditions.

([10]) P. Abrahamson, J. Ben-Arieh, G. Yekutieli and G. Alexander: *Nuovo Cimento*, **12**, 27 (1959).

([11]) P. Ciok, J. Coghen, J. Gierula, R. Hołinsky, A. Jurak, M. Mięsowicz, J. Saniewska and J. Pernegr: *Nuovo Cimento*, **10**, 741 (1958).

([12]) G. Cocconi: *Phys. Rev.*, **111**, 1699 (1958).

The existence of resonant states (pionic isobars) with $m^* \approx 3\,\mu$ and fast two-pion decay has been predicted theoretically for some time ([13]). Identifying tentatively the two « centers » with such isobars we obtain an average transverse momentum of the secondary pions

$$p_T \sim 100 \text{ MeV} . \tag{11}$$

Now, in a « π-π »-collision most of the energy is expected to be transferred to the meson field, and the inelasticity defined as

$$K_p = \frac{3}{2} \langle p_T \rangle \frac{\sum \operatorname{cosec} \theta}{E_0} , \tag{12}$$

where E_0 is the incident energy, would be expected near to unity. Actually our experimental data yield

$$K_p \approx 94\,\% . \tag{13}$$

It is encouraging to observe that the amount of energy transferred to the residual nucleus, as estimated from the mean number of heavy prongs in such collisions, turns out to be

$$(E_h)_{«\pi\text{-}\pi»} = (6.1 \pm 0.4)\,\% E_0 . \tag{14}$$

4. – A similarly high energy transfer to the meson field would be expected in the events pertaining to the « π-3π » peak of Figs. 4 and 5, too. Actually the energy balance, computed under these assumptions, fails to account for $\sim 25\,\%$ of the energy available in the c.m. system. A more likely interpretation of these events is that they are actually « π-2π »-collisions but that the backward isobar is of higher mass, and disintegrates into more particles. Such an interpretation is consistent with the Duller-Walker plot for these events shown in Fig. 8, where the bend is asymmetrically located, favouring a 2:1 backward excess. It can be shown that such an excess is just sufficient to shift the Castagnoli estimate for γ_c from the position corresponding to $\nu = 2$ to $\nu = 3$.

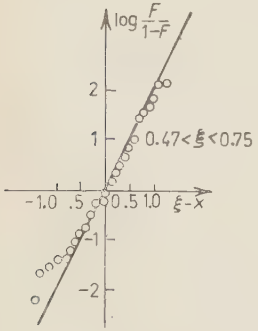


Fig. 8.

([13]) F. Cerulus: *Nuovo Cimento*, **14**, 827 (1959); E. Eberle: *Nuovo Cimento*, **8**, 610 (1958); T. Gotō: *Nuovo Cimento*, **8**, 625 (1958); F. Cerulus and J. von Behr: *Nuovo Cimento*, **16**, 1046 (1960); M. Maximenko: *Žurn. Ėksp. Teor. Fiz.*, **38**, 306 (1960).

Further support for this conclusion comes—in an independent way—from Fig. 9, which shows, separately the spectra of m_t-values for the « π-π » and « π-3π » events as selected according to the ξ-distribution, computed by the method of Birger and Smorodin (²). The position of these peaks, and the average values

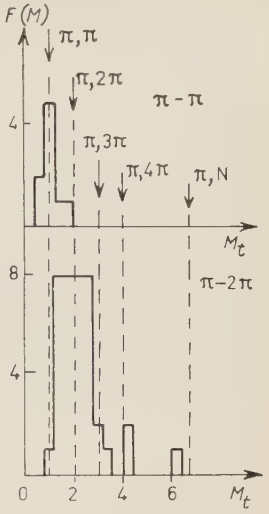


Fig. 9.

$$(\overline{m}_t)_{«\pi\text{-}\pi»} = (1.04 \pm 0.12)\ \mu\,, \tag{15}$$

$$(\overline{m}_t)_{«\pi\text{-}3\pi»} = (2.28 \pm 0.15)\ \mu\,, \tag{16}$$

clearly prove that the effective target masses for the two types of events are consistent with their interpretations as collisions with one, respectively two (more or less) virtual pions in the meson cloud of the target nucleon.

5. – Further proof of the great role played by peripheric collisions with high inelasticity, was obtained by a study of electron pairs from π^0-decay. One hundred nine e^+-e^--pairs (excluding breamsstrahlung ones) were obtained by upstream along the track scanning (¹⁴). The opening angles of all pairs were measured, and the average π^0-energy estimated by the method of the Dubna group (¹⁵). We obtained

$$E_{\pi^0} = (0.87 \pm 0.02)\ \text{GeV} \quad (*). \tag{17}$$

Assuming charge independence, the inelasticity averaged out over all interactions in our stack proves to be

$$\overline{K} = (74 \pm 8)\,\% \tag{18}$$

which is sensibly higher than in the case of proton-induced collisions (~ 33 %) (¹⁵). Assuming for the head-on collisions the same average transverse momentum as in most high-energy nucleonic collisions (¹⁶), we obtain

(*) Statistical errors only.

(¹⁴) D. King: *Phys. Rev.*, **109**, 1344 (1958).

(¹⁵) G. Bayatian, I. Gramenitzky, A. Nomofilov, M. Podgoretzky, and E. Skrzypczak: *Žurn. Èksp. Theor. Fiz.*, **36**, 690 (1959).

(¹⁶) G. B. Zhdanov: *Žurn. Èksp. Theor. Fiz.*, **34**, 856 (1958).

the inelasticity of central collisions as

$$K_c = (65 \pm 2)\,\% \tag{19}$$

and, therefrom the fraction of peripheric collisions

$$P \geqslant 31.5\,\%\,, \tag{20}$$

the inequality sign taking into account a small contribution of head-on collisions to low-multiplicity events. Now, P can also be estimated directly from the ξ-diagrams, which yield

$$P \leqslant 34.1\,\%\,, \tag{21}$$

a result well consistent with eq. (20).

A very similar result, viz. $P \geqslant 27\,\%$ has been obtained in a detailed investigation of peripheric interactions of 9 GeV protons ([17]).

6. – We feel thus justified to conclude that:

a) In the collisions of 7.3 GeV negative pions with emulsion nuclei, a relatively large fraction of the incident energy ($\sim \frac{3}{4}$) is transferred to secondary pions.

b) About one third of the collisions are peripheric ones and can be consistently interpreted as interactions of the incident pion with virtual pions in the meson cloud of the target nucleon.

c) The kinematics of the peripheric meson showers (especially « π-π » events) are consistent with the production and fast decay of pionic resonant (isobaric) states.

* * *

The authors extend their thanks to the Direction of the Joint Institute for Nuclear Research and especially Proff. WANG-KAN-CHANG and V. PETRŽILKA for making available the plates used in this investigation. Our thanks go to Prof. J. AUSLÄNDER, S. ALPER and M. E. MAYER for many stimulating and enlightening discussions. The wholehearted co-operation of Miss I. CALOGHERI in scanning the emulsions and processing the angular data, is gratefully acknowledged.

([17]) E. M. FRIEDLÄNDER: to be published.

RIASSUNTO (*)

Si sono studiate 534 interazioni di pioni negativi a 7.3 GeV in un pacco di emulsioni. Si sono ottenute le molteplicità medie dei rami pesanti e delle particelle dello sciame. Si sono calcolati i fattori individuali di Lorentz γ_c per 395 sciami con $n_s \geqslant 3$. Si è fatto uno studio dettagliato dello spettro γ_c di eventi da nuclei leggeri. Questo spettro risulta composto di collisioni centrali con uno o due nucleoni e collisioni periferiche che coinvolgono uno o due pioni nella nuvola mesonica del nucleone urtato. Le collisioni pione-pione presentano una marcata struttura a due centri, attribuita, in via di tentativo, alla formazione ed al decadimento isotropico in due pioni di stati pionici risonanti (isobarici) di massa $\sim 3\ \mu$. Lo spettro della coppia e^--e^+ da decadimento π^0 è stato misurato e dà una anelasticità media di $\sim 75\%$ in collisioni π^--nucleo.

(*) *Traduzione a cura della Redazione.*

Angular Correlation of Annihilation Radiation in Silica and Aqueous Solutions.

P. Colombino, I. Degregori, L. Mayrone, L. Trossi

Istituto di Fisica dell'Università - Torino

and

S. DeBenedetti (*)

Carnegie Institute of Technology - Pittsburgh, Pa.

(ricevuto il 28 Giugno 1960)

Summary. — We describe here the esperimental apparatus which has been used to study the angular distribution of γ-rays from positrons annihilation. The results of measurements carried out on various forms of silica and on aqueous solutions of several salts are discussed.

1. – Introduction.

The possibility of annihilation of an electron-positron pair into two or more photons, has been predicted theoretically and verified experimentally.

Since it seems established that in condensed matter the positron is slowed down to thermal energy before annihilation, a detailed study of the annihilation process allows the investigation of the state of motion of the negative partner of the annihilation pair. The momentum of the annihilating electron may be revealed through the energy spread of the annihilation γ-rays [1], or through their deviation of exact antiparallelism.

(*) At the Carnegie Institute of Technology positron research is supported by a NSF Grant.

(1) J. W. Dumont, D. A. Lind and B. B. Watson: *Phys. Rev.*, **75**, 1226 (1949).

Because of conservation of momentum the γ-rays will form an angle θ which is related to the component of the momentum perpendicular to the emission direction by

$$\sin\theta = \frac{P_\perp}{mc}.$$

It has been found experimentally that only the peripheral atomic electrons contribute to the annihilation (as it may be expected because of the electrostatic repulsion between the nucleus and the positron); in the case of metals the annihilating electrons are most probably conduction electrons and the annihilation studies have led to a verification of their momentum distribution, according to Fermi statistic.

A phenomenon related to the positron annihilation is the possible formation of a purely electrodynamic atom called positronium whose characteristic mean life is $\approx 10^{-10}$ s in the singlet and $\approx 10^{-7}$ s in the triplet state.

It is thought that positronium may be formed in certain cases also when the positrons annihilate in solid matter. In such cases an anomalously long positron mean life is observed ($\tau_2 \approx 10^{-9}$ s) ([2]).

In the angular distribution curves the presence of positronium is shown by a greater peakdeness of the curve in its central part (narrow component). This is the consequence of the independence of the angular correlation of photons on the electron momenta distribution, since the annihilation positron-electron pair forms a free atom.

Measurements of angular correlation of γ's from two quantum annihilation of positrons in condensed matter have been carried out by some authors ([3]) with the purpose of correlating the amount of narrow component in the angular distribution with the long lifetime τ_2.

The results of the measurements show a relation between narrow components and τ_2 which has been verified in the case of quartz ([3]). A dependence of narrow component on the temperature was also found in the case of teflon ([4,5]).

The evidence of positronium is never found in substances having regular crystal structure as if insufficient space was available for the positronium atom in this case.

Measurements carried out on water and ice ([5]), and on solid and fused naphtalene, suggested that the variation of the narrow component depends on the density of the material rather than on temperature.

([2]) R. E. Bell and R. L. Graham: *Phys. Rev.*, **90**, 644 (1953).

([3]) L. A. Page, M. Heinberg, J. Wallace and T. Trout: *Phys. Rev.*, **98**, 206 (1955).

([4]) A. T. Stewart: *Phys. Rev.*, **99**, 594 (1955).

([5]) R. L. de Zafra and W. T. Joyner: *Phys. Rev.*, **112**, 19 (1958).

The increase of narrow component should then be simply related to an increase in positronium formation as density decreases. On the other hand measurements on the different allotropic states of sulfur recently carried out by us, show no relation between density and narrow component.

In the present work we have studied the angular distribution of two quantum annihilation in the various forms of silica which show differences in density.

We have also carried out a systematic study of aqueous solutions of several salts, with the purpose of investigating chemical properties of positronium.

2. – Experimental apparatus.

The apparatus consists of two 6810 A photomultipliers with plastic scintillators, connected in coincidence and located at a distance of two meters from the sample under study (Fig. 1). The sample is located a few mm under the

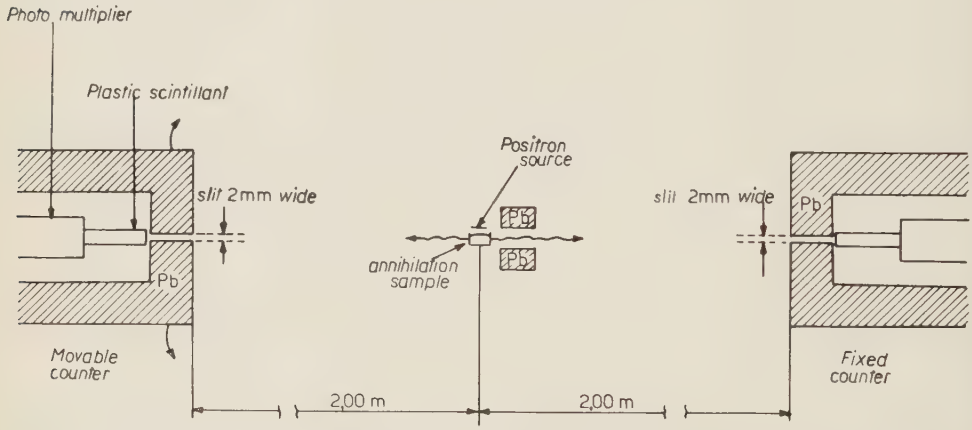


Fig. 1. – Vertical section of the apparatus (schematic). The distance between the source assembly and the detectors is not represented in scale.

positrons source. The source consists of 3 mC of ^{22}Na obtained by evaporating a chloride solution on a thin mica container. This is, in turn, embedded in a lead brick, in order to protect the counters from the direct radiations of the source.

The scintillators are also shielded with lead, forming a linear slit 2 mm wide which subtends an angle of 10^{-3} radians. The sample and the slits are horizontal in order to allow the use of open liquid surfaces as annihilation sources. One of the counters is fixed and the other rotates about the sample.

The displacement of the movable counter is obtained automatically by means of a motor (Fig. 2) controlled by the contacts on the « TIMER » disc (Fig. 3).

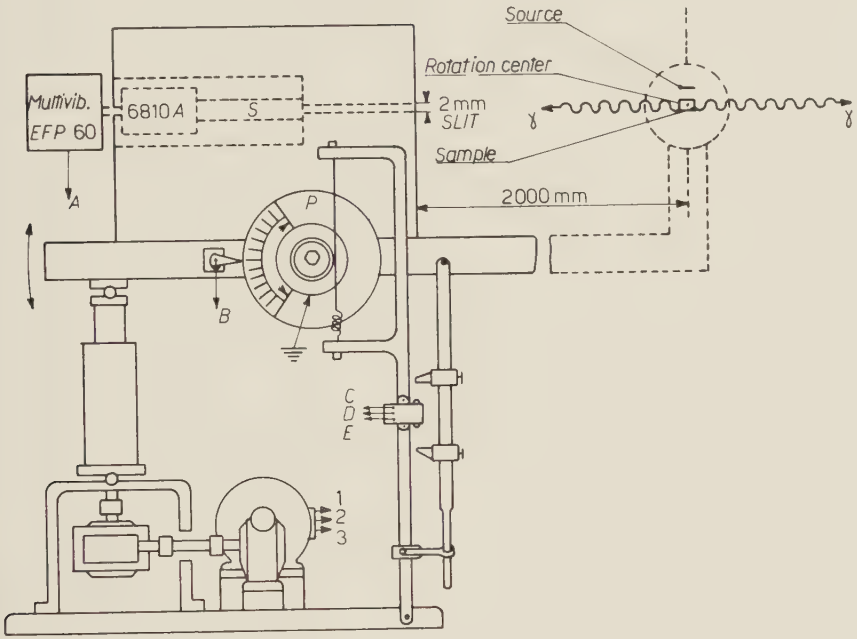


Fig. 2. – Schematic representation of the movable counter.

This disc is rotated by a « clock-work » and carries contacts which ground condenser *C*, causing tyratron 1 to conduct momentarily. Relays 1 and 2 are connected to the plate of this tyratron. Relais 1 closes and keeps tyratron 1 conducting by grounding its grid; it also stops the counting of coincidences through an appropriate gate.

Relais 2 starts the motor through a « tele-inverter » with sufficient delay to insure that the counting is interrupted. It also acts on relais 4 and 3 which control the printing of the data and the resetting of the register.

The motor keeps running until the next contact on disc *P* reaches contact *B* (Fig. 2).

Then tyratron 2 conducts stopping the motor and resetting tyratron 1 in its normal non-conducting state; after this the counting in the new position is started.

Contacts *C*, *D*, *E* (Fig. 2 and 3) belong to a switch which acts on the « tele-inverter », insuring that the motor runs in the appropriate direction.

The electronics consist of an EFP 60 multivibrator on each channel, supplying 10^{-6} s signals. Coincidences with a resolving time of 10^{-7} s are ob-

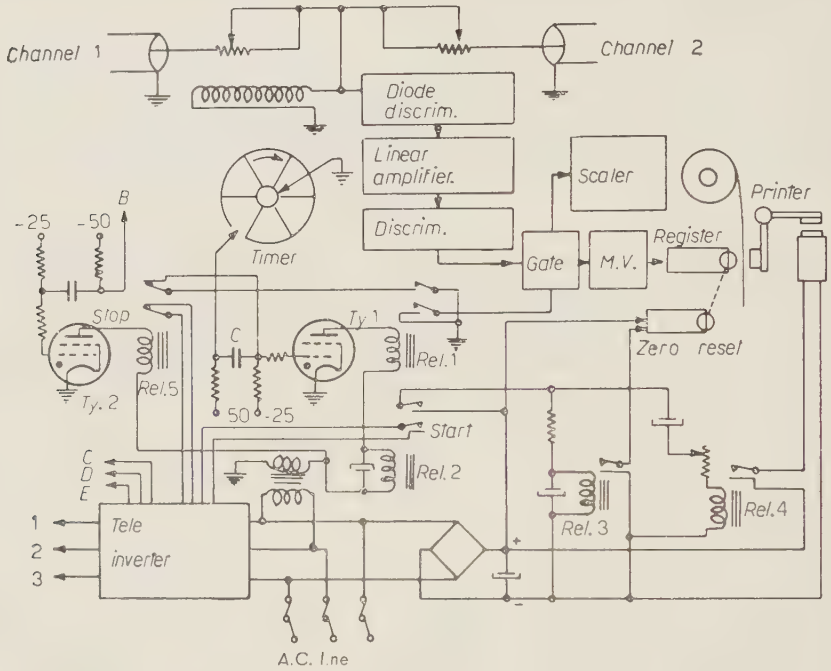


Fig. 3. – Block scheme of the electronic apparatus and automatic registration system.

tained by adding the signals from the two channels on a shorted delay line. After the addition, the pulses are sent to a diode discriminator and to the counting equipment.

3. – Experimental results.

3·1. *Silica.* – Various authors have found a difference in the angular distribution curves obtained with crystal and fused quartz [3]. This difference is in agreement with the idea that ordered crystals do not allow the formation of positronium, since the fused samples exhibit a considerably more prominent peak.

In the case of H_2O, instead, the prominent peaks are observed for ice, and not for liquid water [5].

In order to find an explanation which would be valid both for H_2O and SiO_2, it has been suggested that the « peakdness » of the angular distribution

curves is related to density, rather than to crystal structure. This view would reconcile the behaviour of the two substances, owing to the anomalous behaviour of H_2O near the freezing point.

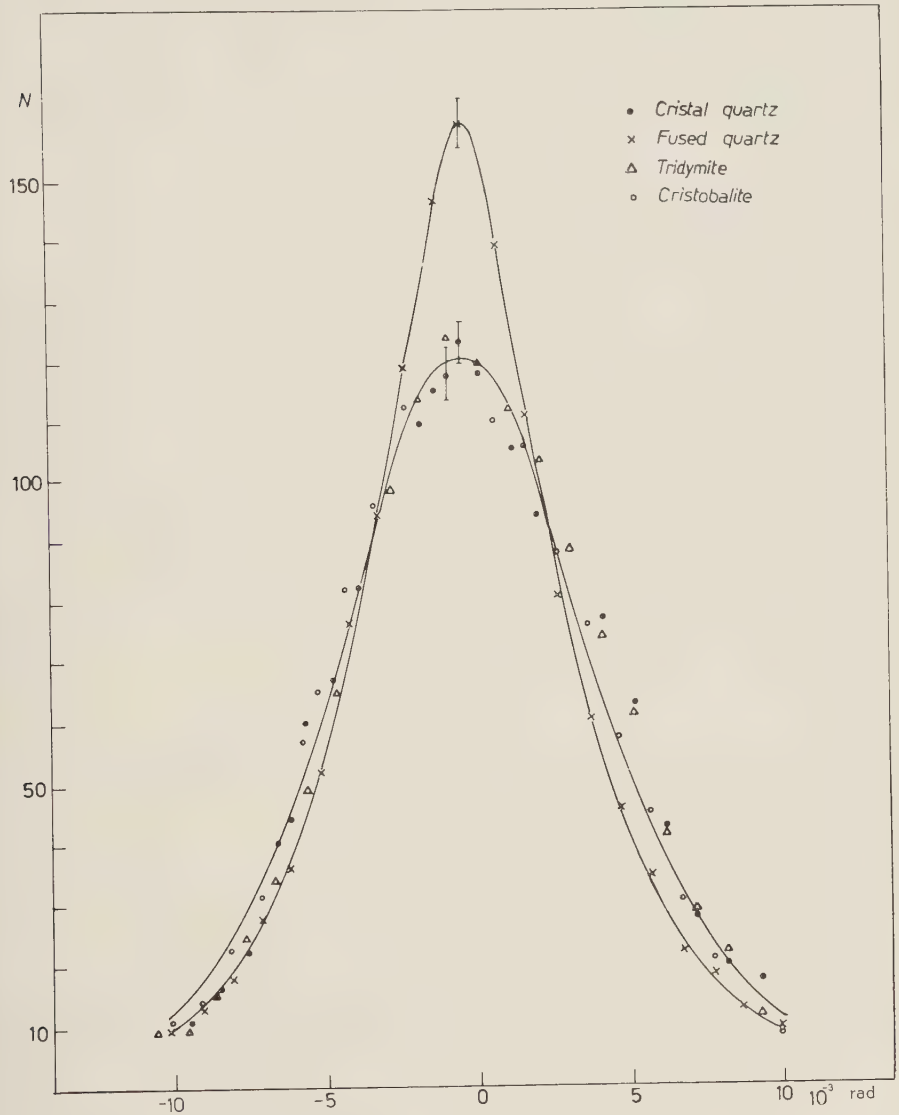


Fig. 4.

In order to obtain further information on this point, we have performed some measurements on different crystal forms of silica. These exhibit appreciable differences of density. In particular, *Tridymite* has a density (2.22),

very close to that of fused quartz, and a crystal structure similar to that of ordinary ice. The silicon atoms, in tridymite, lie at the positions in the crystal that are occupied by the oxygen atoms in ice ([6]).

Cubic ice, formed by condensation of water vapour at temperatures around that of liquid air, is closely similar in structure to the cubic form of silica Cristobalite.

Some authors think of the existence in liquid water of hydrogen-bonded aggregates with a structure resembling that of quartz which is denser than tridymite although the interatomic distances are the same.

We have procured our samples of Tridymite and Cristobalite from the «Clinica del Lavoro» Luigi Devoto, University of Milan.

They prepared Trydimite by heating quartz at 1350 °C for 25 days. The Cristobalite was supplied by the same Institute. Both samples were used in powder form.

The results are plotted in Fig. 4 and are all normalized to equal area. The curve of fused quartz (density: $2.1 \div 2.3$) shows an enhancement on the top with respect to that of crystal quartz (density: 2.65), as was already pointed out by PAGE *et al.* ([3]). This is in accord with the differences in density and with the presence of a long lifetime component τ_2 in fused quartz.

The other crystal forms of silica do not show any narrow component even though their density is very closed to that of fused quartz. Such results seem to suggest that a density effect may not be considered the only reason for positronium formation, and is in agreement with our previous measurements carried out on the allotropic states of sulfur.

3·2. *Aqueous solutions.* – Data on the behaviour of positrons in aqueous solutions are interesting because of their possible connection with the chemical behaviour of the «element» positronium.

It is known for instance that solutions of paramagnetic salts show shorter mean life ([7]) and larger peakedness ([8]) than pure water, and that nitric solutions show shorter mean life ([9]) and smaller peakedness ([8]).

Recently MCGERVEY and DEBENEDETTI ([10]) have started a systematic study of aqueous solutions with the three-quantum-annihilation technique. They have confirmed with this method that paramagnetic salts and nitric solutions destroy positronium, and found in hydrochloric and sulfuric solutions

---

([6]) L. PAULING: *The Structure of Water*, in D. HADZI and H. W. THOMPSON: *Hydrogen Bounding* (New York).

([7]) R. E. GREEN and R. E. BELL: *Can. Journ. Phys.*, **36**, 1684 (1958).

([8]) R. L. DE ZAFRA: *Phys. Rev.*, **113**, 1547 (1959).

([9]) R. E. GREEN and R. E. BELL: *Can. Journ. Phys.*, **35**, 398 (1957).

([10]) J. MCGERVEY and S. DEBENEDETTI: *Phys. Rev.*, **114**, 495 (1959).

some indication that positronium may replace metallic ions which are lower than Zn in the electrochemical scale. In such a manner they determined the position of the element positroniun near Zn.

We have repeated and extended the previous measurements on angular correlation from solutions in order to attempt a clarification of these points.

The results (*) obtained with paramagnetic salts (Fig. 5) confirm the increase in the narrow component, as expected from the conversion of triplet positronium into singlet.

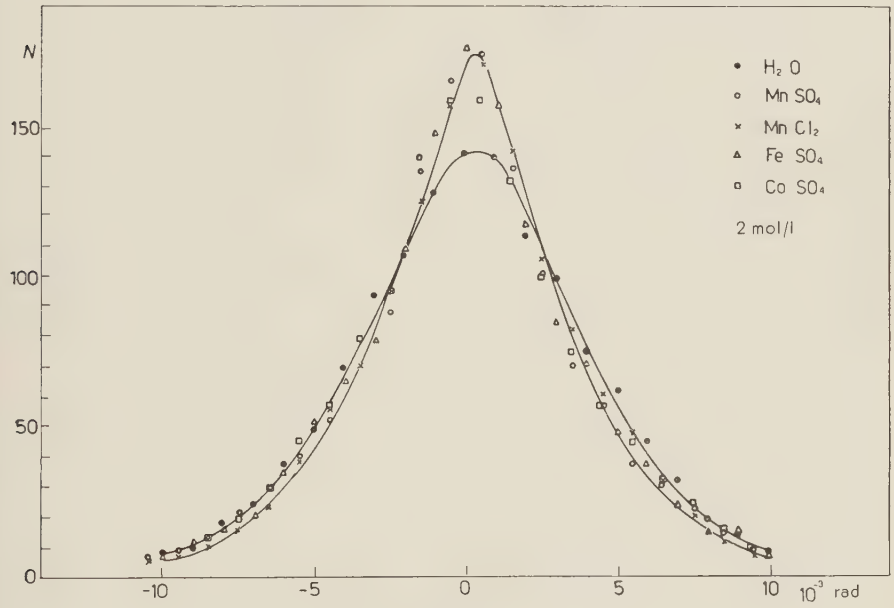


Fig. 5.

The addition of the radical NO_3, instead produces the opposite effect (Fig. 6, 7) indicating that the triplet positronium is not destroyed by conversion into singlet, but by some other process, and possibly by a reaction of chemical oxidation, as it has been suggested.

The increase of the narrow component seems indipendent from the particular nitrate used, and to depend only on the concentration of the NO_3^- ion ([8]).

Finally, we were unable to find any effect with chlorides (Fig. 8, 9) and sulfates of any metal (Fig. 10-11).

(*) All curves are normalized to the same area.

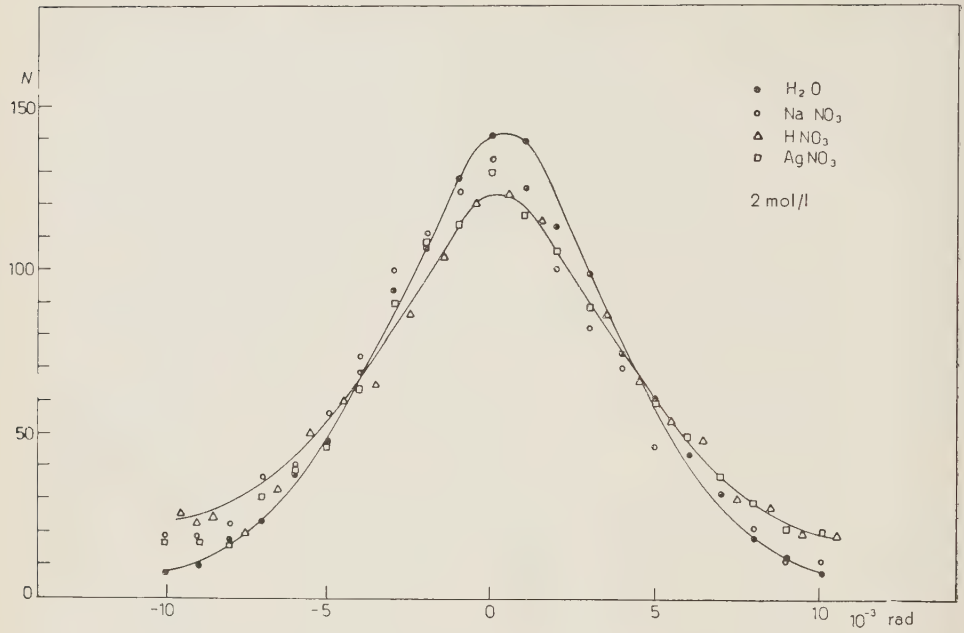


Fig. 6.

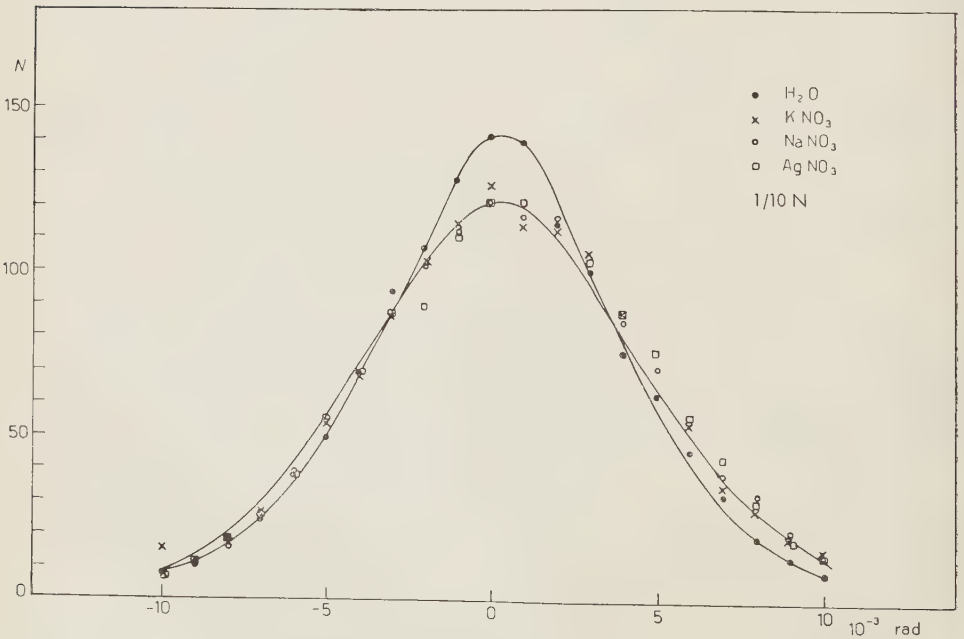


Fig. 7.

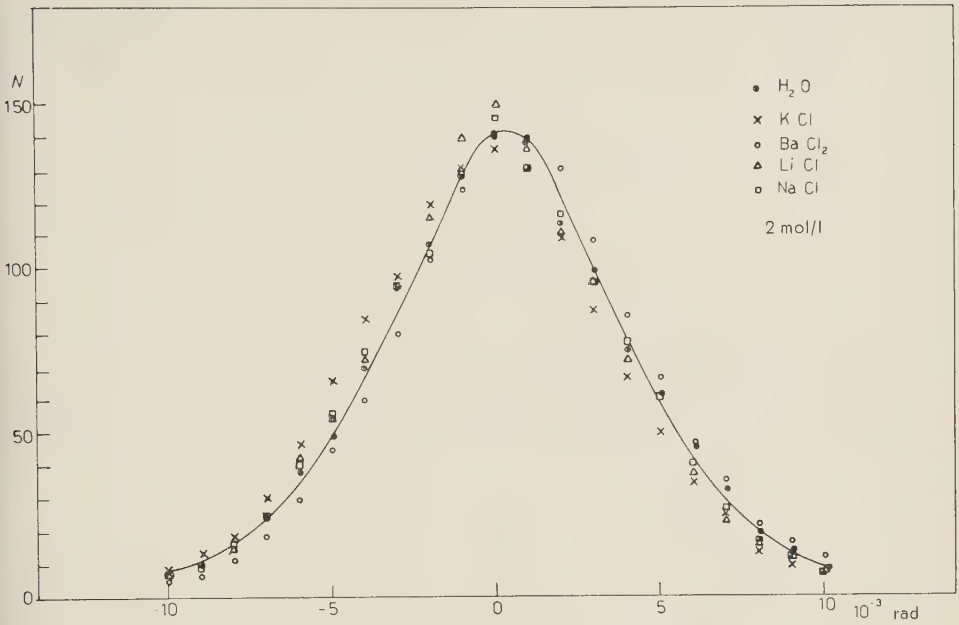


Fig. 8.

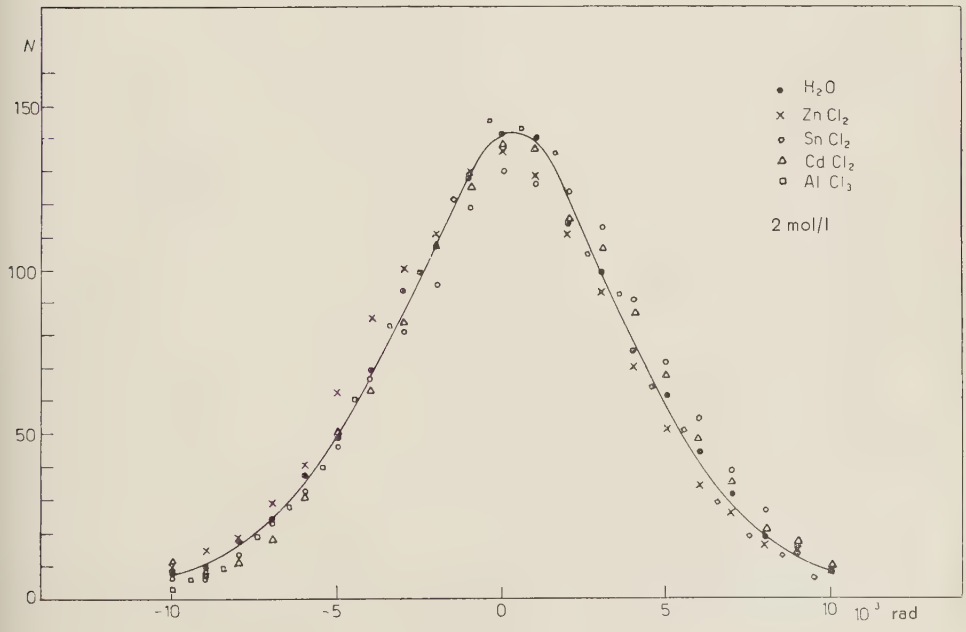


Fig. 9.

3865

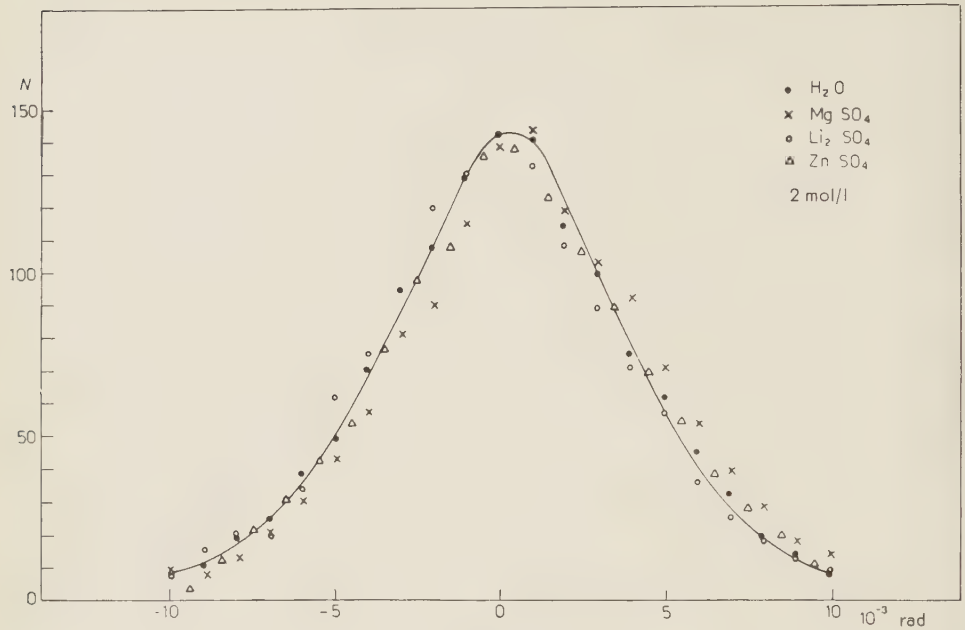


Fig. 10.

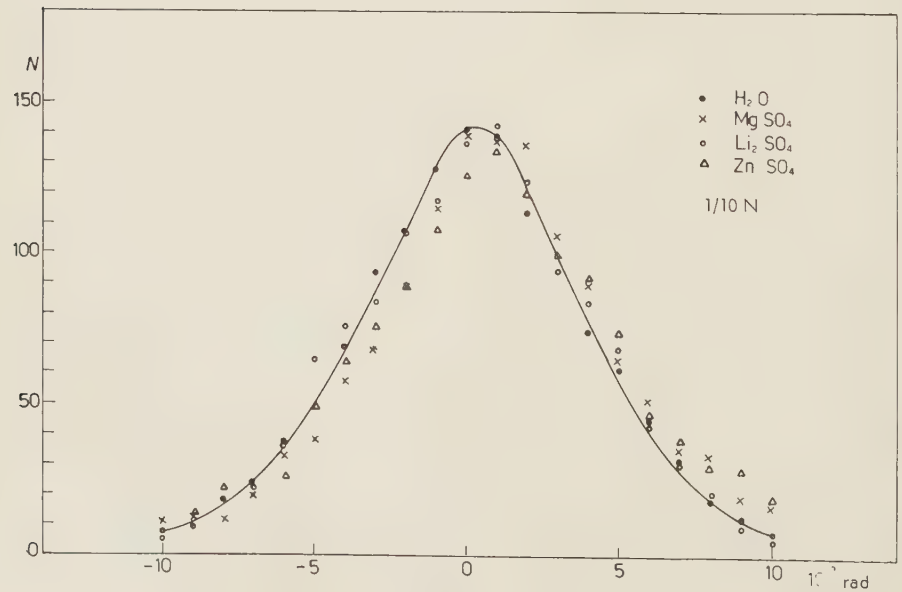


Fig. 11.

This does not necessarily contradict that some metallic ions are replaced by positronium, because such process could occur without significant changes on the angular distribution curves.

RIASSUNTO

Si descrive il dispositivo sperimentale montato allo scopo di studiare la distribuzione angolare dei raggi γ di annichilamento dei positoni. Sono discussi i risultati relativi a misure eseguite su diverse forme di silice, e su soluzioni acquose di diversi sali.

Covariant Formalism for Particle Dynamics (*).

A. Peres

Israel Institute of Technology - Haifa

N. Rosen (**)

Israel Institute of Technology - Haifa
Weizmann Institute of Science - Rehovoth

(ricevuto il 30 Giugno 1960)

Summary. — In the usual covariant formulations of particle dynamics difficulties are encountered that appear to be associated with the constraint imposed of the four-velocity, $u_k u^k = 1$. It is proposed to remove these difficulties by putting the Lagrangian into a suitable form, so that this condition on the velocities is a consequence of equations of motion together with appropriate initial conditions, rather than a constraint.

---

1. – Covariant Lagrangian and Hamiltonian formalisms for particles in various fields have recently been the subject of rather involved investigations (1-4). Difficulties seem to be associated with the non-holonomic constraint $u^k u_k = 1$, which is imposed on the four-velocity (5).

If this constraint is taken into account by the method of Lagrange mul-

---

(*) Partly supported by the U.S. Air Force through the European Office of the Air Research and Development Command.

(**) Permanent address: Israel Institute of Technology.

(1) L. Infeld: *Bull. Acad. Polon. Sci.*, **5**, 491 (1957).

(2) G. Szamosi: *Geneva Conference*, P/1604 (1958).

(3) F. Sauter: *Zeits. f. Phys.*, **156**, 275 (1959).

(4) G. Kalman: preprint (1960).

(5) Indices run from 0 to 3. We take

$$ds^2 = g_{mn}\,dx^m\,dx^n = dt^2 - dx^2 - dy^2 - dz^2, \qquad \text{and} \qquad u^k = dx^k/ds\,.$$

tipliers [1], the equations of motion obtained from

$$\delta \int L(x^k, u^k)\, \mathrm{d}s = 0\,, \tag{1}$$

are

$$\mathrm{d}p_k/\mathrm{d}s = \partial L/\partial x^k\,, \tag{2}$$

where [3]

$$p_k = [L - u^n(\partial L/\partial u^n)]u_k + (\partial L/\partial u^k)\,. \tag{3}$$

These equations do not have the canonical Euler-Lagrange form, and therefore cannot lead to a Hamiltonian formalism [4].

On the other hand, if we do not take into account the constraint, $u^k u_k = 1$ the equations of motion do have the canonical form, and it may occur that they satisfy the constraint. For instance, for a free particle, one may choose

$$L = \tfrac{1}{2} m(u^k u_k - 1)\,, \tag{4}$$

whence

$$p_k = \partial L/\partial u^k = m u_k\,, \tag{5}$$

$$\mathrm{d}p_k/\mathrm{d}s = \partial L/\partial x^k = 0\,. \tag{6}$$

In this case it follows from (5) and (6) that $u^k u_k$ is a constant, which can be equated to unity by suitable initial conditions. However, in most cases the equations of motion derived in such a way will be incompatible with $u^k u_k = 1$.

2. – It may appear that one can get rid of the constraint by introducing, instead of $\mathrm{d}s$, a new invariant parameter $\mathrm{d}\tau$, which is left completely arbitrary. One then has

$$\mathrm{d}s = (w^k w_k)^{\frac{1}{2}}\, \mathrm{d}\tau\,, \tag{7}$$

where $w^k = \mathrm{d}x^k/\mathrm{d}\tau$, and

$$u^k = w^k (w^n w_n)^{-\frac{1}{2}}\,. \tag{8}$$

The new variational principle is

$$\delta \int L'(x^k, w^k)\, \mathrm{d}\tau = 0\,, \tag{9}$$

where

$$L'(x^k, w^k) = L[x^k, w^k (w^n w_n)^{-\frac{1}{2}}] (w^s w_s)^{\frac{1}{2}} . \tag{10}$$

The Lagrange equations of motion now have the canonical form. It is easily verified that

$$p_k = \partial L'/\partial w^k \tag{11}$$

is equal to the p_k defined by eq. (3).

However, it follows from eq. (10) that L' is homogeneous of the first order in the w^k, so that p_k is homogeneous of the zero-th order in the w^k. This fact has two undesirable consequences: the Hamiltonian

$$H' = p_k w^k - L' \tag{12}$$

vanishes identically, and moreover, it is not possible to solve (11) for the w^k in terms of the p_k and x^k. The latter is associated with the fact that in (11) one can eliminate the w^k and obtain a relation between the p_k and x^k. Obviously, this new constraint is equivalent to the previous one. Let us write it

$$K(x^k, p_k) = 0 . \tag{13}$$

If we desire to work in terms of momenta, rather than of velocities, we must therefore use Dirac's generalized Hamiltonian dynamics [6]. In our case, the equations of motion of any dynamical variable will be

$$\mathrm{d}F/\mathrm{d}\tau = A[F, K] , \tag{14}$$

where A is arbitrary. The arbitrariness of A is obviously related to the arbitrariness of τ. The evolution of the dynamical system is nevertheless unambiguous, because time-derivatives are independent of A:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}F/\mathrm{d}\tau}{\mathrm{d}t/\mathrm{d}\tau} = \frac{[F, K]}{[t, K]} . \tag{15}$$

If desired, it is possible to determine A so that $\mathrm{d}\tau = \mathrm{d}s$. For the sake of brevity, let

$$L - u^k(\partial L/\partial u^k) = M . \tag{16}$$

---

[6] P. A. M. Dirac: *Proc. Roy. Soc.*, A **246**, 326 (1958).

One has

$$\partial M/\partial u^k = -u^n(\partial^2 L/\partial u^k\,\partial u^n)\,, \tag{17}$$

whence

$$\partial p_k/\partial u^n = g_{kn}\,M + (\delta^s_k - u_k u^s)(\partial^2 L/\partial u^n\,\partial u^s)\,. \tag{18}$$

It follows that

$$u^k(\partial p_k/\partial u^n) = M u_n\,. \tag{19}$$

Multiplying both sides by $(\partial u^n/\partial p_s)$, one obtains

$$u^s = M u_n(\partial u^n/\partial p_s) = \tfrac{1}{2} M[\partial(u^n u_n - 1)/\partial p_s]\,. \tag{20}$$

If this is to be one set of canonical equations, it may be inferred that $A = \frac{1}{2}M$. This assumption may be verified by computing

$$-\tfrac{1}{2} M[\partial(u^n u_n - 1)/\partial x^k] = -M u_n(\partial u^n/\partial x^k)_p\,, \tag{21}$$

$$= M u_n(\partial u^n/\partial p_s)_x(\partial p_s/\partial x^k)_u\,, \tag{22}$$

$$= u^s(\partial p_s/\partial x^k)_u\,, \tag{23}$$

where use has been made of (19). Now

$$u^s(\partial p_s/\partial x^k) = u^s[u_s(\partial M/\partial x^k) + (\partial^2 L/\partial x^k\,\partial u^s)]\,. \tag{24}$$

A straightforward computation, making use of (16), leads to

$$u^s(\partial p_s/\partial x^k) = \partial L/\partial x^k = \mathrm{d}p_k/\mathrm{d}s\,, \tag{25}$$

thus confirming that if $A = \frac{1}{2}M$, one has $\mathrm{d}\tau = \mathrm{d}s$.

These results, which are formally equivalent to those of SAUTER [3], are not quite satisfactory: in some steps of the derivation, we treated the four u^k as independent variables, while in others, we made explicit use of $u^k u_k = 1$. Such *ad hoc* procedures should be avoided, if possible.

As a matter of fact, the use of Poisson brackets is also not free from ambiguities, as their definition implies the independence of the p_k and x^k.

3. – It would be much more preferable if the constraint $u^k u_k = 1$ followed from the equations of motion, as we have seen in the case of a free particle.

In other words, one should choose the Lagrangian in such a way that $u^k u_k$ be a constant of motion, *whatever may be the potential.*

Now, if the Lagrangian does not explicitly depend on the path parameter, as in our case, the Hamiltonian is such a constant (provided there are no constraints). Thus, one should choose the Lagrangian in such a way that the Hamiltonian be a function of $u^k u_k$ only. This is easily performed by modifying L' in the following way:

$$L' \to L' + m[\tfrac{1}{2} w^k w_k - (w^k w_k)^{\frac{1}{2}}] . \tag{26}$$

It follows

$$p_k \to p_k + m w_k [1 - w^n w_n)^{-\frac{1}{2}}] . \tag{27}$$

Thus, the p_k are no more homogeneous of zero-th order in the w^k, and one can solve (27) for the w^k as function of the p_k and x^k. Further, one has

$$H' \to \tfrac{1}{2} m w^k w_k , \tag{28}$$

so that $w^k w_k$ is a constant, which may be equated to unity by a suitable choice of initial conditions. One has then $\mathrm{d}\tau = \mathrm{d}s$, and the change in p_k as given by (27) vanishes. The situation here is analogous to the canonical formulation of electromagnetic theory, in which the Lorentz condition follows from the equations of motion with suitable initial conditions.

In practice it may be more convenient to follow a somewhat different, but essentially equivalent, procedure. Any particle Lagrangian L can be written in the form

$$L = L_0 + L_i , \tag{29}$$

where L_0, the free particle Lagrangian, is given by

$$L_0 = \tfrac{1}{2} m u_k u^k , \tag{30}$$

and L_i is the interaction Lagrangian. The procedure consists in making L_i a homogeneous function of u^k of degree one. This is done by replacing in L_i, u^k by $u^k (u_n u^n)^{-\frac{1}{2}}$ and then multiplying L_i by $(u_n u^n)^{\frac{1}{2}}$, in analogy with what is indicated in eq. (10). Assuming that there are no constraints, one finds that

$$H = \tfrac{1}{2} m u_k u^k , \tag{31}$$

where u_k is to be expressed as a function of p_n and x^n. The constancy of H, which follows from the equations of motion, then leads to the possibility of obtaining $u_k u^k = 1$ by choosing suitable initial conditions.

Alternatively, in place of (30) one can take

$$L_0 = \tfrac{1}{2} m(u_k u^k - 1) . \tag{32}$$

This leads to

$$H = \tfrac{1}{2} m(u_k u^k + 1) , \tag{33}$$

which, for $u_k u^k = 1$, becomes

$$H = m , \tag{34}$$

so that the Hamiltonian in this formalism is equal to the rest-energy of the particle.

4. – As an example, we shall derive the Hamiltonian of a particle in a combined electromagnetic and scalar field. The Lagrangian is to be taken:

$$L = \tfrac{1}{2} m u^k u_k + e A_k u^k + V(u^k u_k)^{\frac{1}{2}} . \tag{35}$$

One obtains

$$p_k = \partial L / \partial u^k = m u_k + e A_k + V u_k (u^n u_n)^{-\frac{1}{2}} , \tag{36}$$

whence

$$p_k - e A_k = [m + V(u^n u_n)^{-\frac{1}{2}}] u_k , \tag{37}$$

$$(p^k - e A^k)(p_k - e A_k) = [m(u^k u_k)^{\frac{1}{2}} + V]^2 . \tag{38}$$

It follows

$$H = \tfrac{1}{2} m u^k u_k = \{[(p^k - e A^k)(p_k - e A_k)]^{\frac{1}{2}} - V\}^2 / 2m . \tag{39}$$

From this Hamiltonian, one reobtains

$$u^k = \partial H / \partial p_k = (p^k - e A^k)\{\ \} / m[(p^n - e A^n)(p_n - e A_n)]^{\frac{1}{2}} . \tag{40}$$

The requirement $u^k u_k = 1$ gives

$$m = \{[(p^k - e A^k)(p_k - e A_k)]^{\frac{1}{2}} - V\} , \tag{41}$$

whence

$$u^k = (p^k - e A^k)/(m + V) . \tag{42}$$

Finally

$$\mathrm{d}p_k/\mathrm{d}s = -\partial H/\partial x^k = u^n A_{n,k} + V_{,k}\,, \tag{43}$$

where use has been made of (41) and (42).

* * *

The writers would like to thank Prof. G. Szamosi and Dr. A. B. Volkov for helpful discussions.

RIASSUNTO (*)

Nelle usuali formulazioni covarianti della dinamica delle particelle si incontrano difficoltà che sembrano associate alla costrizione imposta sulla quadri-velocità, $u_k u^k = 1$. Si propone di rimuovere queste difficoltà scrivendo il Lagrangiano in forma opportuna, in modo che questa condizione per le velocità sia una conseguenza delle equazioni del moto associate ad appropriate condizioni iniziali, piuttosto che di una costrizione.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVIII, N. 4 16 Novembre 1960

Pion-Kaon Scattering.

M. Gourdin

Faculté des Sciences - Orsay (*) *and Bordeaux*

Y. Noirot and Ph. Salin

Faculté des Sciences - Bordeaux (**)

(ricevuto il 9 Luglio 1960)

Résumé. — Une interaction directe méson π-méson K a été proposé comme étant l'un des processus possibles responsable des interactions méson K-nucléon. Nous essayons dans ce papier de voir ce que la représentation de Mandelstam, dans la forme simplifiée et realiste donnée par Cini et Fubini, permet de determiner d'une telle interaction. Nous donnons une technique générale de calcul pour les différentes ondes partielles et calculons la contribution d'une intéraction directe méson π-méson π à l'amplitude de diffusion méson π-méson K. Enfin, faisant l'hypothèse d'une onde S importante, nous donnons pour cette dernière une expression dans l'approximation de la portée effective et évaluons l'importance des corrections dues au croisement possible des mésons π. L'étude des ondes P n'est pas abordée dans ce papier et est en cours d'investigation.

1. – Introduction.

1·1. – The possibility of a direct $\pi\pi$-KK interaction has been proposed a few years ago by Yamaguchi [1,2] and Barshay [3,4] in order to explain experimental data on K-meson-nucleon elastic scattering, K^- absorption by pro-

(*) Postal address: Laboratoire de Physique Théorique et Hautes Energies, B.P. 12, Orsay, Seine et Oise.

(**) Postal address: Service de Physique Théorique, Chemin Brunet Talence, Gironde.

(1) Y. Yamaguchi: *Proc. of Padova-Venice Conference on Recently Discovered Particles* (1957), chap. v, p. 42.

(2) Y. Yamaguchi: *Suppl. Progr. Theor. Phys.*, no. 11, 37 (1959).

(3) S. Barshay: *Phys. Rev.*, **109**, 2160 (1958).

(4) S. Barshay: *Phys. Rev.*, **110**, 743 (1958).

tons, associated production of strange particles. By considering the following types of graphs.

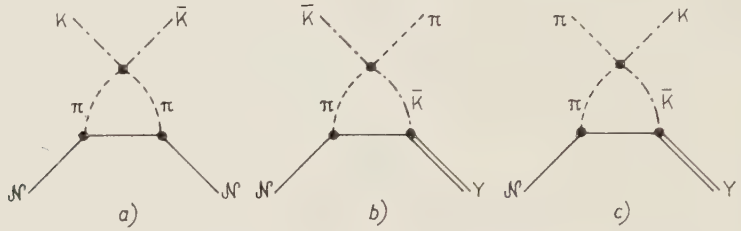


Fig. 1. – – – – refers to a π-meson; —— refers to a nucleon; – · – · – refers to a K-meson; ═══ refers to an hyperon (Λ, Σ).

they are able to reproduce some features of the experimental results one cannot obtain with the conventional perturbation graphs

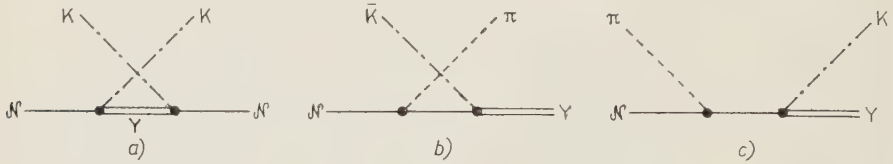


Fig. 2.

Another interesting process where can enter a direct $\pi\pi$-$K\bar{K}$ interaction is the production of a π-meson in K-meson-nucleon collisions. Experimentally the following reactions have been observed:

π^--meson production ([5-7])

$$K^+ + n \to K^+ + p + \pi^-$$

$$K^+ + p \to K^0 + p + \pi^+$$

π^+-meson production ([7,8])

$$K^+ + n \to K^0 + n + \pi^+ .$$

([5]) E. HELMY, J. H. MULVEY, D. J. PROWSE and D. H. STORK: *Phys. Rev.*, **112**, 1793 (1958).

([6]) B. SECHI-ZORN and G. T. ZORN: *Bull. Am. Phys. Soc.*, **3**, 24 (1958).

([7]) W. F. FRY: Seminar at CERN (June 1960).

([8]) M. GRILLI, L. GUERRIERO, M. MERLIN, Z. O'FRIEL and G. SALANDIN: *Nuovo Cimento*, **2**, 358 (1958).

The relatively important ratio

$$R = \frac{\sigma(\mathrm{K}+\mathrm{p} \to \mathrm{K}+\mathrm{p}+\pi)}{\sigma(\mathrm{K}+\mathrm{p} \to \mathrm{K}+\mathrm{p})} ,$$

cannot be explained by perturbation calculation ([9]) and suggests the possibility of an important contribution due to the mechanism.

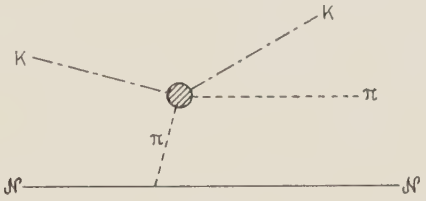


Fig. 3.

But an important final interaction between the π-meson and the nucleon (state $I - J = \frac{3}{2}$) as given by an isobaric model can also increase the ratio R. Preliminary experimental results ([7]) give

$$\frac{\sigma(\mathrm{K}^+ + \mathrm{n} \to \mathrm{K}^0 + \mathrm{n} + \pi^+)}{\sigma(\mathrm{K}^+ + \mathrm{p} \to \mathrm{K}^0 + \mathrm{p} + \pi^+)} \simeq 5/32 ,$$

where the isobaric model with only charge independence relations predicts $\frac{1}{6}$. On the other hand, if we assume a strong $I = \frac{3}{2}$, $J = 1$ π-K interaction we obtain the same theoretical result by considering only the graph given in Fig. 3. The study of the angular correlations between the three final particles is the only test to decide if the $\mathrm{K}\bar{\mathrm{K}}\pi\pi$ interaction can play or not an important role in the production of a π-meson in K-meson-nucleon collisions.

1'2. – The aim of this present work is to try to apply the Mandelstam formalism to the π-K problem. In a previous paper ([10]) we give the general structure of the equations for the π-K scattering and the $\pi + \pi \to \mathrm{K} + \bar{\mathrm{K}}$ reaction by restricting ourselves to the case of S wave. It is shown that the application of the unitarity condition leads us in the $\pi + \pi \to \mathrm{K} + \bar{\mathrm{K}}$ channel to Omnes-Mushkelishvili equations one may easily solve after definition of left hand cuts. For the scattering channels, the integral equations are non linear and of the Low type.

([9]) C. Ceolin, N. Dallaporta and L. Taffara: *Nuovo Cimento*, **9**, 353 (1958).
([10]) M. Gourdin, Y. Noirot and Ph. Salin: *Diffusion méson π-méson* K (Bordeaux, Dec. 1959), unpublished.

In the first two parts of this paper (Sections **2** and **3**), we develop a general formalism to define partial wave amplitudes for fixed values of isotopic spin I and angular momentum J.

In Section **4** we study the reaction $\pi+\pi \to \mathrm{K}+\overline{\mathrm{K}}$ and retain only the state $I=J=1$ corresponding to a two-pion resonance giving the form of its contribution to π-K scattering for the case of a narrow resonance.

We then obtain in Section **5** a set of non-linear integral equations for the partial scattering amplitudes, coupled by crossing.

One particular type of solution with a dominant S wave is found in Section **6** for these equations. An effective range formula can be given and it is shown that the corrections due to the crossed terms are quite important. The effect of a pion-pion interaction is discussed.

2. – Invariance properties.

2·1. – Let us call q_1 and q_2 the ingoing pions four momenta and p_1 and p_2 the corresponding ones for the K-mesons

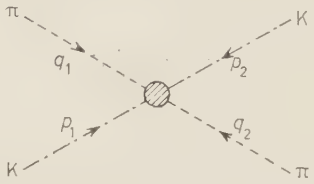


Fig. 4.

We define, as usual, three scalar quantities

$$s_1 = -(q_1+p_1)^2\,, \qquad s_2 = -(q_1+p_2)^2\,, \qquad t = -(q_1+q_2)^2\,,$$

with the relation on the energy shell

$$s_1+s_2+t = 2(\mu^2+\varkappa^2)\,,$$

where μ is the π-meson mass and $\varkappa$ the K-meson mass. The three channels are defined as follows:

Channel I	π-K	scattering	$(p_1)+(q_1) = (-p_2)+(-q_2)$
Channel II	π-$\overline{\mathrm{K}}$	scattering	$(p_2)+(q_1) = (-p_1)+(-q_2)$
Channel III	K-$\overline{\mathrm{K}}$	pair production	$(q_1)+(q_2) = (-p_1)+(-p_2)$

and we can write the S-matrix element in the form

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta_4(p_1 + q_1 + p_2 + q_2) \frac{1}{(2\pi)^6} \frac{1}{4(p_1^0 p_2^0 q_1^0 q_2^0)^{\frac{1}{2}}} T_{fi} . \tag{1}$$

2·2. – The problem of the isospin dependence of T is the same as for the π-meson-nucleon scattering. If α and β are the isospin indices for the ingoing and outgoing π-meson, we can consider T as a matrix in the isospin space of the K-meson. The two well-known invariants are $\delta_{\beta\alpha}$ and $\frac{1}{2}[\tau_\beta, \tau_\alpha]$ and we can put

$$T_{\beta\alpha} = \delta_{\beta\alpha} T^{(0)} + \tfrac{1}{2}[\tau_\beta, \tau_\alpha] T^{(1)} , \tag{2}$$

$T^{(0)}$ and $T^{(1)}$ are two scalar functions of s_1, s_2, t.

In the channel III, the total isospin can be 0 and 1, and the corresponding eigen-amplitudes are $T^{(0)}$ and $T^{(1)}$. In channels I and II, the two possibilities are $I = \frac{1}{2}$ and $I = \frac{3}{2}$; if we call $T^{(\frac{1}{2})}$ and $T^{(\frac{3}{2})}$ the amplitudes in these two states, the connection with $T^{(0)}$ and $T^{(1)}$ is given by

$$T^{(\frac{1}{2})} = T^{(0)} + 2T^{(1)} , \qquad T^{(\frac{3}{2})} = T^{(0)} - T^{(1)} . \tag{3}$$

2·3. – We can cross the two pions; this leads to the transformation

$$\alpha \leftrightarrow \beta , \qquad s_1 \leftrightarrow s_2 , \qquad t \leftrightarrow t . \tag{4}$$

The first isospin invariant is symmetrical and the second antisymmetrical in the exchange on the two pions. Consequently we obtain for the scalar functions $T^{(0)}$ and $T^{(1)}$ the simple properties:

$$T^{(0)}(s_2, s_1, t) = T^{(0)}(s_1, s_2, t) ,$$

$$T^{(1)}(s_2, s_1, t) = - T^{(1)}(s_1, s_2, t) .$$

We will study only two independent channels: the scattering and the pair production.

2·4. *Channel I.* – In the center of mass system we call $\boldsymbol{k}$ and $\boldsymbol{k}'$ the π-mesons relative momenta and θ the scattering angle:

$$p_1 = (-\boldsymbol{k}, e_k) \qquad q_1 = (\boldsymbol{k}, \omega_k) \qquad \omega_k = \sqrt{\mu^2 + k^2} , \qquad e_k = \sqrt{\varkappa^2 + k^2} ,$$

$$p_2 = (\boldsymbol{k}', -e_k) \qquad q_2 = (-\boldsymbol{k}', -\omega_k) \qquad k = k' , \qquad \boldsymbol{k} \cdot \boldsymbol{k}' = k^2 \cos\theta .$$

The scalar invariants are function of k and θ only

$$(5)\qquad \begin{cases} s_1 = (e_k + \omega_k)^2 = W^2\,, \\ s_2 = (e_k - \omega_k)^2 - 2k^2(1 + \cos\theta)\,, \\ t \;\; = -2k^2(1 - \cos\theta)\,, \end{cases}$$

where W is the total energy in the center of mass system. The scattering amplitude is a function of W and $\cos\theta$ and we define

$$(6)\qquad T^{(I)}(s_1, s_2, t) = G^{(I)}(W, \cos\theta) \qquad I = \tfrac{1}{2}, \tfrac{3}{2}$$

for the two isospin states.

We now expand $G^{(I)}(W, \cos\theta)$ in Legendre polynomial of $\cos\theta$

$$(7)\qquad G^{(I)}(W, \cos\theta) = \sum_J (2J+1)\, G_J^{(I)}(W)\, P_J(\cos\theta)$$

with the reciprocal formula

$$(8)\qquad G_J^{(I)}(W) = \frac{1}{2}\int_{-1}^{+1} G^{(I)}(W, x)\, P_J(x)\, \mathrm{d}x\,.$$

The differential cross-section is given, in terms of partial amplitudes $G_J^{(I)}(W)$ by

$$\frac{\mathrm{d}\sigma_I}{\mathrm{d}\Omega} = \frac{1}{(8\pi W)^2}\,|\sum_J (2J+1) G_J^I(W)\, P_J(\cos\theta)\,|^2\,,$$

and the total cross-section

$$\sigma_I(W) = \frac{4\pi}{(8\pi W)^2}\sum_J (2J+1)\,|G_J^{(I)}(W)\,|^2\,.$$

Below the threshold corresponding to the production of one supplementary meson, we can define a real phase shift $\bar{\delta}_J^{(I)}(W)$ for each partial scattering amplitude, related to $G_J^{(I)}(W)$ by

$$(9)\qquad G_J^{(I)}(W) = \frac{8\pi W}{k}\exp\,[i\bar{\delta}_J^{(I)}]\,\sin\bar{\delta}_J^{(I)},$$

and we obtain immediately the classical formula for the differential and total cross-sections in terms of phase shifts.

In this region the unitarity condition takes the simple form

$$\text{Im}\, G_J^{(I)}(W) = \frac{k}{8\pi W} |G_J^I(W)|^2 . \tag{10}$$

For the other values of W, we retain only the $(\pi + \text{K})$ intermediate state in the unitarity condition and we assume the relation to be valid in the entire range of W, neglecting the contributions due to inelastic processes.

2·5. *Channel III.* – In the center of mass system we use $\boldsymbol{q}$ and $\boldsymbol{p}$ for the π-meson and K-meson relative momenta

$$\begin{aligned} q_1 &= (\ \ \boldsymbol{q}, \omega) & p_1 &= (-\boldsymbol{p}, -\omega) & \omega &= \omega_q = e_p \\ q_2 &= (-\boldsymbol{q}, \omega) & p_2 &= (+\boldsymbol{p}, -\omega) & \boldsymbol{p}\cdot\boldsymbol{q} &= pq\cos\varphi . \end{aligned}$$

The three scalar quantities s_1, s_2, t can be written

$$\begin{aligned} s_1 &= -(p^2 + q^2) + 2pq\cos\varphi , \\ s_2 &= -(p^2 + q^2) - 2pq\cos\varphi , \\ t &= 4\omega^2 = 4(q^2 + \mu^2) = 4(p^2 + \varkappa^2) . \end{aligned}$$

For the reaction amplitude, we adopt the notation

$$T^{(I)}(s_1, s_2, t) = F^{(I)}(t, \cos\varphi) \qquad I = 0, 1 \tag{11}$$

We expand $F^{(I)}(t, \cos\theta)$ in Legendre polynomial of $\cos\varphi$,

$$F^{(I)}(t, \cos\varphi) = \sum_J (2J + 1)\, F_J^{(I)}(t)\, P_J(\cos\varphi) , \tag{12}$$

$$F_J^{(I)}(t) = \frac{1}{2}\int_{-1}^{+1} F^{(I)}(t, y)\, P_J(y)\, \mathrm{d}y . \tag{13}$$

We easily see that $\cos\varphi$ is odd in the exchange of two pions. It follows that for $I = 0$, only even values of J occur and for $I = 1$ only odd value of J.

In the expression of the unitarity of the S-matrix, we retain only the two-pion states and we obtain the well-known relation for each partial amplitude

$$\text{Im} F_J^{(I)}(t) = F_J^{(I)}(t) h_J^{(I)*}(t) , \tag{14}$$

where $h_J^{(I)}(t) = \exp[i\delta_J^{(I)}] \sin\delta_J^{(I)}$ is the pion-pion scattering amplitude in the state of isospin I and angular momentum J and $\delta_J^{(I)}$ the corresponding phase shift.

3. – Mandelstam representation.

3·1. – We assume for the two scalar functions $T^{(0)}$ and $T^{(1)}$ a Mandelstam representation [11]. It is easy to verify that Cini and Fubini arguments [12] can be applied in order to reduce the two dimensional representation to a one dimensional representation. We can write

$$T^{(I)}(s_1, s_2, t) = f^{(I)}(s_1, Z_1) + \varepsilon_I f^{(I)}(s_2, Z_2) + g^{(I)}(t, Y) \qquad I = 0, 1, \tag{15}$$

with $\varepsilon_0 = 1$, $\varepsilon_1 = -1$ by crossing.

The variables Z_1, Z_2, Y are arbitrary combinations of s_1, s_2, t which do not reduce respectively to s_1, s_2, t. The function $f^{(I)}(s, Z)$ possesses a strong dependence in the variable s and a weak dependence in the variable Z; the same result holds for $g^{(I)}(t, Y)$ in the strong variable t and the weak variable Y. It follows that we expand $f^{(I)}(s, Z)$ and $g^{(I)}(t, Y)$ in the variables Z and Y.

$$\begin{cases} f^{(I)}(s, Z) = \sum_J (2J+1)\, a_J^{(I)}(s)\, P_J(Z) \\ g^{(I)}(t, Y) = \sum_J (2J+1)\, b_J^{(I)}(t)\, P_J(Y) \end{cases} \tag{16}$$

and assume for the coefficients $a_J^{(I)}(s)$ and $b_J^{(I)}(t)$ a spectral representation explicitly given later.

It is convenient to choose for Z_1 the cosine of the scattering angle in the channel I, given by

$$Z_1 = 1 + \frac{2s_1 t}{[s_1 - (\varkappa + \mu)^2][s_1 - (\varkappa - \mu)^2]}, \tag{17}$$

the corresponding one for Z_2 in the channel II and for Y the value $\cos\varphi$ introduced in the channel III

$$Y = \frac{s_1 - s_2}{[(t - 4\mu^2)(t - 4\varkappa^2)]^{\frac{1}{2}}}. \tag{18}$$

3·2. *Analyticity of the expansions.* – The expansion of $f^{(I)}(s_1, Z_1)$ in Legendre polynomials of Z_1 introduces in the channels II and III singularities

[11] S. MANDELSTAM: *Phys. Rev.*, **112**, 1344 (1958).
[12] M. CINI and S. FUBINI: *Ann. Phys.*, **3**, 352 (1960).

corresponding to the poles of Z_1. If we consider the range of variation for s_1 in the three channels, we obtain the results reproduced in the Fig. 5.

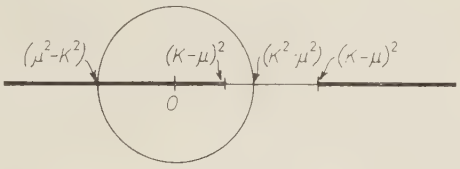


Fig. 5.

Channel I cut from $(\varkappa+\mu)^2$ to $+\infty$

Channel II cut from $(\varkappa-\mu)^2$ to $-\infty$

Channel III
- cut from $(\mu^2-\varkappa^2)$ to $-\infty$, physical region $t>4\varkappa^2$
- circle of center O and radius $\varkappa^2-\mu^2$, unphysical region $4\mu^2<t<4\varkappa^2$

The pole $s_1=(\varkappa-\mu)^2$ of Z_1 (eq. (17)) is inside the range of variation; consequently the expansion of $f^{(I)}(s, Z)$ in Legendre polynomials $P_J(Z)$ introduces spurious singularities. In order to remove these later, we assume for $a_J^{(I)}(s)$ the following spectral representation

$$a_J^{(I)}(s)=[s-(\varkappa+\mu)^2]^J[s-(\varkappa-\mu)^2]^J\frac{1}{\pi}\int_{\varkappa^2+\mu^2}^{\infty}\frac{\tau_J^{I}(s')\,ds'}{s'-s\quad i\varepsilon}. \tag{19}$$

With this form, we have automatically the correct low energy behaviour for the phase shifts: $\bar{\delta}_l\simeq k^{2l+1}$ (except for $\bar{\delta}_0$ for which the condition will be formulated in an other manner).

The situation is not the same for the function $g^{(I)}(t, Y)$. The range of variation for t is trivial to determine:

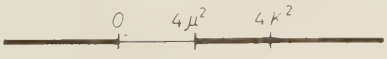


Fig. 6.

Channel I and II cut from 0 to $-\infty$

Channel III
- cut from $4\mu^2$ to $4\varkappa^2$ unphysical region
- cut from $4\varkappa^2$ to $+\infty$ physical region

and we have no spurious singularities introduced by Y in the physical region.

We assume for $b_J^{(I)}(t)$ the spectral representation

$$b_J^{(I)}(t) = (pq)^J \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{\sigma_J^{(I)}(t')}{(p'q')^J (t' - t - i\varepsilon)} \, dt' . \tag{20}$$

3·3. *Unitarity conditions.* – By inspection of the range of variation for s_1, s_2, t, it is easy to see that

$$\operatorname{Im} T^{(I)}(s_1, s_2, t) = \begin{cases} \operatorname{Im} f^{(I)}(s_1, Z_1) & \text{for channel I} \\ \varepsilon_I \operatorname{Im} f^{(I)}(s_2, Z_2) & \text{for channel II} \\ \operatorname{Im} g^{(I)}(t, Y) & \text{for channel III} \end{cases}$$

If we now compare the expansions (7), (12), (16), we deduce

$$\operatorname{Im} G_J^{(I)}(W) = \operatorname{Im} a_J^{(I)}(s_1) \qquad I = \tfrac{1}{2}, \tfrac{3}{2}$$

$$\operatorname{Im} F_J^{(I)}(t) = \operatorname{Im} b_J^{(I)}(t) \qquad I = 0, 1$$

and taking into account the relations (10) and (14) we have

$$\begin{cases} \operatorname{Im} a_J^{(I)}(s) = \dfrac{k}{8\pi W} |G_J^{(I)}(W)|^2 , & I = \tfrac{1}{2}, \tfrac{3}{2}, \\ \operatorname{Im} b_J^{(I)}(t) = F_J^{(I)}(t) h_J^{(I)*}(t) . & I = 0, 1 . \end{cases} \tag{21}$$

The weight functions $\tau_J^{(I)}(s)$ and $\sigma_J^{(I)}(t)$ occuring in the spectral representation (19) and (20) are determined by the unitarity conditions (21). We finally obtain for $a_J^{(\frac{1}{2})}(s)$ and $a_J^{(\frac{3}{2})}(s)$

$$a_J^{(I)}(s) = [S - (\varkappa + \mu)^2]^J [S - (\varkappa - \mu)^2]^J \frac{1}{\pi} \int_{(\varkappa+\mu)^2}^{\infty} \frac{1}{[s' - (\varkappa + \mu)^2]^J [s' - (\varkappa - \mu)^2]^J} \cdot \frac{k'}{8\pi W'} \frac{|G_J^{(I)}(W')|^2}{s' - s - i\varepsilon} \, ds' , \tag{22}$$

for $b_J^{(0)}(t)$ and $b_J^{(1)}(t)$

$$b_J^{(I)}(t) = (pq)^J \frac{1}{\pi} \int_{(2\mu)^2}^{\infty} \frac{F_J^{(I)}(t') h_J^{(I)*}(t')}{t' - t - i\varepsilon} \frac{1}{(p'q')^J} \, dt' . \tag{23}$$

4. – Integral equations in channel III.

4·1. – We use the function:

$$H^{(I)}(t, \cos\varphi) = f^{(I)}(s_1, Z_1) + \varepsilon_I f^{(I)}(s_2, Z_2) \qquad I = 0, 1$$

to describe the contributions from the π-K scattering to the $K\bar{K}$ pair production. In this channel, $H^{(I)}(t, \cos\varphi)$ is a real function of the two variables t and $\cos\varphi$.

The partial amplitude of angular momentum J, satisfies a Mushkelishvili-Omnès integral equation,

$$\frac{F_J^I(t)}{(pq)^J} = f_J^{(I)}(t) = \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{f_J^{(I)}(t')\, h_J^{(I)*}(t')}{t' - t - i\varepsilon}\, \mathrm{d}t' + H_J^{(I)}(t)\,, \tag{24}$$

where

$$H_J^{(I)}(t) = \frac{1}{2}\int_{-1}^{+1} \frac{H^{(I)}(t, y)\, P_J(y)}{(pq)^J}\, \mathrm{d}y\,.$$

The introduction of a left hand cut from $-\infty$ to $-\mu^2$ for the variable t [10] leads to an integral representation of $H_J^{(I)}(t)$

$$H_J^{(I)}(t) = \frac{1}{\pi}\int_{-\infty}^{-\mu^2} \frac{\mathrm{Im}\, f_J^{(I)}(z)}{t - z}\, \mathrm{d}z\,. \tag{25}$$

With this form for the unhomogeneous terms, the Mushkelishvili-Omnès equation has a simple solution [13]

$$f_J^{(I)}(t) = \exp\left[\varrho_J^{(I)}(t) + i\delta_J^{(I)}(t)\right] \cdot \left[\frac{1}{\pi}\int_{-\infty}^{-\mu^2} \frac{\mathrm{Im}\, f_J^{(I)}(z)\, \exp\left[-\varrho_J^{(I)}(z)\right]}{t - z}\, \mathrm{d}z + \mathscr{P}_J^{(I)}(t - 4\mu^2)\right], \tag{26}$$

[13] R. Omnès: *Nuovo Cimento*, **8**, 316 (1958); M. Gourdin and A. Martin: *Nuovo Cimento*, **17**, 224 (1960), App. III.

where $\varrho_J^{(I)}(t)$ is defined from the π-π phase shift as usual

$$\varrho_J^{(I)}(t) = P\,\frac{t}{\pi}\int_{(2\mu)^2}^{\infty}\frac{\delta_J^{(I)}(t')}{t'(t'-t)}\,\mathrm{d}t'\,,$$

with a subtraction at $t=0$.

The function $\mathscr{P}_J^{(I)}(t-4\mu^2)$ corresponds to the general solution of the associated homogeneous integral equation and essentially depends on the behaviour of the corresponding π-π phase shift $\delta_J^{(I)}(t)$.

With the form (26), the $K\bar{K}$ pair production amplitude can be written

$$(27)\qquad F^{(I)}(t,\cos\varphi) = \sum_J (2J+1)\exp\left[\varrho_J^{(I)}(t) + i\delta_J^{(I)}(t)\right](pq)^J\cdot$$
$$\cdot\left[\frac{1}{\pi}\int_{-\infty}^{-\mu^2}\frac{\mathrm{Im}\,f_J^{(I)}(z)\exp\left[-\varrho_J^{(I)}(z)\right]}{t-z}\,\mathrm{d}z + \mathscr{P}_J^{(I)}(t-4\mu^2)\right]P_J(\cos\varphi)\,.$$

If the π-π amplitude is zero, in a I, J, state, we evidently have

$$\delta_J^{(I)}(t)\equiv 0\,,\qquad \exp\left[\varrho_J^{(I)}(t)\right]\equiv 1\,,\qquad \mathscr{P}_J^{(I)}(t-4\mu^2)\equiv 0$$

and the corresponding $K\bar{K}$ pair production is only determined by π-K scattering contributions, $H_J^{(I)}(t)$.

4·2. – This last remark permits us to know directly the form of the equations in channel I and II by analytic continuation of equation (27). Because we have now $t<0$, the π-π phase shift $\delta_J^{(I)}(t)$ is zero and the contribution $g^{(I)}(t,Y)$ from $K\bar{K}$ pair production to π-K scattering is easily written in the form

$$(28)\qquad g^{(I)}(t,Y) = \sum_J (2J+1)\left\{\exp\left[\varrho_J^{(I)}(t)\right]\mathscr{P}_J^{(I)}(t-4\mu^2) +\right.$$
$$\left.+\frac{1}{\pi}\int_{-\infty}^{-\mu^2}\frac{\mathrm{Im}\,f_J^{(I)}(z)}{t-z}\left[\exp\left[\varrho_J^{(I)}(t)-\varrho_J^{(I)}(z)\right]-1\right]\mathrm{d}z\right\}P_J\left[\frac{s_1-s_2}{\sqrt{(t-4\mu^2)(t-4\varkappa^2)}}\right](pq)^J.$$

The unknown function $\mathrm{Im}\,F_J^{(I)}(z)$ is given in terms of π-K scattering amplitude by equation (25) after a left hand cut introduction.

4·3. – We now keep only the $I=J=1$ two-pion contribution assuming a sharp resonance in this state. We replace the cut over t: $(2\mu)^2<t<\infty$ by a pole at $t=t_r$ in order to reproduce, in the scattering channels ($t<0$), the

effect of such a resonance. For $\exp[\varrho_1^{(1)}(t)]$ we can use with a good approximation the following form [14]

$$\exp[\varrho_1^{(1)}(t)] = \frac{t_r}{t_r - t} . \qquad t < 0 .$$

The integral over z, occuring in equation (28) can be transformed into

$$\frac{1}{\pi}\int_{-\infty}^{-\mu^2} \frac{\operatorname{Im} f_1^{(1)}(z)}{t - z}\left[\exp[\varrho_1^{(1)}(t) - \varrho_1^{(1)}(z)] - 1\right] dz \simeq \frac{1}{t_r - t}\frac{1}{\pi}\int_{-\infty}^{-\mu^2} \operatorname{Im} f_1^1(z)\, dz .$$

The last integral, describing the contributions due to π-K scattering is a constant I we can evaluate later. On the other hand the homogeneous Omnès-Mushkelishvili integral equation in the case $I = J = 1$ possesses the solution $\lambda \exp[\varrho_1^{(1)}(t) + i\delta_1^{(1)}(t)]$ corresponding to a value λ for $\mathscr{P}_1^{(1)}$ [13]. We finally obtain for $g^{(1)}(t, Y)$ the simple form

$$g^{(1)}(t, Y) = \frac{C(s_1 - s_2)}{t_r - t} . \tag{29}$$

With the aid of equation (3), we easily deduce

$$g^{\frac{1}{2}} = 2\, g^{(1)} , \qquad g^{\frac{3}{2}} = - g^{(1)} . \tag{30}$$

We can note that such a form is equivalently given by the « bipion model » as is explained in similar papers on nucleon-nucleon scattering [15] and photoproduction of pions on nucleons [16].

4·4. – In order to take into account the $I = J = 0$ two-pion state we can describe in the low energy limit the π-π scattering in the corresponding state by a scattering length [17,18]. Such a correction is not considered in this paper.

[14] M. Gourdin and A. Martin: *Nuovo Cimento*, **16**, 78 (1960).

[15] D. Amati, E. Leader and B. Vitale: *Nuovo Cimento*, **18**, 409 (1960).

[16] M. Gourdin, D. Lurié and A. Martin: *Effect of a pion-pion scattering resonance on low energy meson-photoproduction* (CERN, June 1960, to be published in *Nuovo Cimento*).

[17] G. F. Chew, S. Mandelstam and H. P. Noyes: UCRL-9001.

[18] N. H. Khuri and S. B. Treiman: Princeton preprint.

5. – Integral equations in channel I.

5·1. – It is convenient to consider only eigen states for the isospin: in this section, the index (I) refers only to the values $\frac{1}{2}$ and $\frac{3}{2}$. The crossing condition (4) becomes more complicated and we put

$$G^{(I)}(W, \cos\theta) = f^{(I)}(s_1, Z_1) + \overline{f}^{(I)}(s_2, Z_2) + g^{(I)}(t, Y) \,. \tag{31}$$

The relation (3) gives

$$\overline{f}^{(I)}(s, Z) = \sum_{I'} \lambda_{II'} f^{(I')}(s, Z) \,, \tag{32}$$

where the crossing matrix is given by

$$\lambda = \begin{matrix} & \begin{matrix} \frac{1}{2} & \frac{3}{2} \end{matrix} \\ \begin{matrix} \frac{1}{2} \\ \frac{3}{2} \end{matrix} & \begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix} \end{matrix} .$$

The expansion of $f^{(I)}(s, Z)$ in Legendre polynomials $P_J(Z)$ leads us to define a set of functions $\overline{a}_J^{(I)}(s)$ given, in terms of $a_J^{(I)}(s)$ in the same manner as $\overline{f}^{(I)}(s, Z)$ in terms of $f^{(I)}(s, Z)$ (eq. (32)). We finally obtain for the partial wave amplitude $G_J^{(I)}(W)$ the following expression

$$G_J^{(I)}(W) = a_J^{(I)}(s_1) + \frac{1}{2}\int_{-1}^{+1} \Big[P_J(\cos\theta)\, \mathrm{d}\cos\theta \cdot \tag{33}$$

$$\cdot \Big\{ \sum_L (2L+1)\, \overline{a}_L^{(I)}(s_2)\, P_L\left(1 + \frac{2s_2 t}{[s_2 - (\varkappa+\mu)^2][s_2 - (\varkappa-\mu)^2]}\right) + g^{(I)}(t, Y) \Big\} \Big] \,,$$

where s_1, s_2, t, Y are function of W and $\cos\theta$ given by the relations (5) and (18); $a_J^{(I)}(s_1)$ and $\overline{a}_L^{(I)}(s_2)$ are related to the partial scattering amplitudes by the integral representation (22).

In conclusion, we have to solve a system of non linear coupled integral equations, the coupling occuring between the values of isospin and angular momentum by crossing and contributions from $K\overline{K}$ pair production.

5·2. *The subtraction problem.* – We have not considered the problem of convergence for our integral representations. On the other hand we have not introduced any coupling constant related to the intensity of the π-K interaction. Following Chew and Mandelstam [19] we take, as a parameter, the

[19] G. F. Chew and S. Mandelstam: UCRL-8806.

value of the reaction amplitude at an arbitrary no physical point. As it was pointed out by OKUBO ([20]) there is some ambiguity in the choice of such a point. Nevertheless if we require the crossing symmetry, we obtain

$$s_1 = s_2 = s_0 , \qquad t_0 = 2(\varkappa^2 + \mu^2 - s_0)$$

and by inspection of Fig. 5, we immediately deduce

$$(\varkappa^2 - \mu^2) < s_0 < (\varkappa^2 + \mu^2) .$$

We then can define the subtraction constant by

$$a_0 = T^{(0)}(s_0, s_0, t_0) .$$

and because of the crossing symmetry $T^{(1)}(s_0, s_0, t_0) = 0$. It follows that the two π-K scattering amplitudes $T^{(\frac{1}{2})}$ and $T^{(\frac{3}{2})}$ are equal at this unphysical point.

Another alternative is to assume $s_1 = -s_2$, it follows a unique value $2(\varkappa^2+\mu^2)$ for t_0 and an arbitrary value for s_1 and s_2 which yields this possibility doubtful.

6. – *S*-wave dominant solution for π-K scattering.

We now only retain the S wave as important for the π-K scattering amplitude. The system (33) reduces to the following set of equations

$$\text{(34)} \qquad \begin{cases} G_0^{(I)}(W) = a_0^{(I)}(s_1) + \dfrac{1}{2}\displaystyle\int_{-1}^{+1} \mathrm{d}\cos\theta \Big[\sum_{I'} \lambda_{II'} a_0^{(I')}(s_2) + g^{(I)}(t,\ Y)\Big] \\ G_l^{(I)}(W) = \dfrac{1}{2}\displaystyle\int_{-1}^{+1} \mathrm{d}\cos\theta\ \Big[\sum_{I'} \lambda_{II'} a_0^{(I')}(s_2) + g^{(I)}(t,\ Y)\Big] P_l(\cos\theta) . \end{cases}$$

The first equation (34) is an integral equation of the Low type and the others permit to deduce the real l-wave for the scattering amplitude, from the knowledge of the S-wave.

6·1. – If we neglect, for the moment all contributions due to the left hand cut, we obtain a simple integral equation for $G_0^{(I)}(W)$

$$\text{(35)} \qquad G_0^{(I)}(W) = a_0 + \frac{s - s_0}{\pi} \int_{(\varkappa+\mu)^2}^{\infty} \frac{k'}{8\pi W'} \frac{|G_0^{(I)}(W')|^2}{(s' - s - i\varepsilon)(s' - s_0)} \mathrm{d}s' .$$

([20]) S. OKUBO: *Phys. Rev.*, **118**, 357 (1960).

By introducing the reciprocal function $g_0^{(I)}(s_1) = 8\pi/G_0^{(I)}(W)$ we immediately determine the imaginary part of $g_0^{(I)}(s_1)$ by unitarity

$$\text{Im}\, g_0^{(I)}(s) = -\frac{k}{W}\,. \tag{36}$$

The analytical properties of $g_0^{(I)}(s)$ in the complex plane of the variable s, deduced from those of $G_0^{(I)}(W)$ permit us to obtain an explicit expression for $g_0^{(I)}(s)$ ([21])

$$g_0(s) = \frac{8\pi}{a_0} - \frac{s - s_0}{\pi}\int_{(\varkappa+\mu)^2}^{\infty} \frac{k'}{W'}\,\frac{\mathrm{d}s'}{(s'-s_0)(s'-s-i\varepsilon)}\,. \tag{37}$$

The isospin index (I) has been dropped in $g_0(s)$ because at this stage we have no difference between the states $I = \frac{1}{2}$ and $I = \frac{3}{2}$. The last integral can be easily performed and if we put

$$l(s) = \frac{1}{\pi}\left[\frac{\varkappa^2-\mu^2}{2s}\log\frac{\varkappa}{\mu} + \right. \tag{38}$$

$$\left. + \frac{\sqrt{[s-(\varkappa+\mu)^2][s-(\varkappa-\mu)^2]}}{2s}\log\frac{s-(\varkappa^2+\mu^2)+\sqrt{[s-(\varkappa+\mu)^2][s-(\varkappa-\mu)^2]}}{2\varkappa\mu}\right],$$

we can write equation (37) in the simple form:

$$g_0(s) = \begin{cases} \dfrac{8\pi}{a_0} + l(s) - l(s_0) - i\,\dfrac{k}{W}\,, & \text{for } s \geqslant (\varkappa+\mu)^2, \\ \dfrac{8\pi}{a_0} + l(s) - l(s_0)\,. & \text{for } s \leqslant (\varkappa+\mu)^2. \end{cases} \tag{39}$$

From equation (39) we are able to write, in the physical region for the π-K scattering, an effective range formula for the S phase shift without any difference between the two isospin states

$$\frac{k}{W}\,\mathrm{ctg}\,\bar{\delta}_0 = \frac{8\pi}{a_0} + l(s) - l(s_0)\,. \tag{40}$$

This solution is not unique: in particular, if $G_0^{(I)}(W)$ goes to zero, it corresponds for $g_0(s)$ a pole we must include in the form (39). A detailed discussion of equations of Low has been given by CASTILLEJO, DALITZ and DYSON ([22])

([21]) G. FELDMAN, P. T. MATTHEWS and A. SALAM: *Nuovo Cimento*, **16**, 549 (1960).
([22]) L. CASTILLEJO, R. DALITZ and F. DYSON: *Phys. Rev.*, **101**, 453 (1956).

and we consider here the most simple type of solution, without any experimental information.

6·2. – Equation (40) has been previously given by OKUBO [20] using a perturbation theory with the conventional point interaction hamiltonian:

$$H_{\pi K} = 4\pi\lambda_K(\boldsymbol{\varphi}_\pi \boldsymbol{\varphi}_\pi)(\varphi_K^* \varphi_K)\,, \tag{41}$$

where φ_π is the isovector π-meson field and φ_K the isospinor K-meson field. By summing only chain diagrams we can derive a formula identical to (40). The subtraction constant is simply the coupling constant $8\pi\lambda_K = -a_0$.

Fig. 7.

With such a hamiltonian (41) we cannot obtain an isospin splitting because the two pions and the two kaons can only be in a $I = 0$ state. At all orders the $T^{(1)}$ amplitude is zero and consequently $T^{(\frac{1}{2})} = T^{(\frac{3}{2})}$. But in our formalism we have no reason *a priori* to neglect the isospin flip part of the scattering amplitude and to forbid the charge exchange processes. The distinction between the state $I = \frac{1}{2}$ and $I = \frac{3}{2}$ which does not appear on the approximate solution (40), can be made if we include the left hand cut contributions due to the channel III. We shall go back on this point later.

6·3. – We can see in equation (38) that the function $l(s)$ is a relatively smooth in the unphysical range $(\varkappa^2 - \mu^2) < s_0 < \varkappa^2 + \mu^2$. The arbitrariness in the choice of the convenient subtraction point is really not important. A numerical evaluation gives

$$0.30 < l(s_0) < 0.32$$

and we take an average value $l(s_0) = 0.31$.

If we forbid any resonance for the S phase shift (*) at low energy (**) we

(*) See footnote [20] of reference [19].

(**) The application of such a solution as (40) in the high energy domain is evidently a nonsense, because we have neglected intermediate state with one kaon and three pions in the unitarity condition.

can deduce an allowed range of variation for the constant a_0, simply related to the S scattering length (*)

$$-1 < \frac{a_0}{8\pi} < 5 \,. \qquad \text{for } (\varkappa+\mu)^2 \leqslant s < 4(\varkappa+\mu)^2 . \tag{42}$$

6·4. – In order to take into account the crossed diagrams it can be introduced an iteration procedure (19). One see on equation (34) that crossing corrections can be expressed in terms of an integral involving the previous solution. More precisely we have obtained a solution in the form

$$\frac{1}{8\pi} G_0(W) = \left(M(s) - i \frac{k}{W} \right)^{-1}, \tag{43}$$

where

$$M(s) = \frac{8\pi}{a_0} + l(s) - l(s_0) \,, \tag{44}$$

and the corrected solution can be written as

$$\frac{1}{8\pi} G_0(W) = \left(M(s) - i \frac{k}{W} \right)^{-1} + \frac{1}{2} \int_{-1}^{+1} \mathrm{d} \cos\theta \left| \frac{1}{M(s_2)} \right| - a_0 \,. \tag{45}$$

In the same approximation one can directly calculate the total crossing term contribution from the S-wave solution (43) and we may write

$$\frac{1}{8\pi} G_0(W, \cos\theta) = \left[M(s) - i \frac{k}{W} \right]^{-1} + \frac{1}{M(s_2)} - a_0 \,. \tag{46}$$

6·5. – We have computed the corrections due to the crossed term for various values of $\cos\theta$, W and a_0 by comparing the differential cross-sections one can obtain from equations (46) and (43). Fig. 8 gives the ratio

$$R = \frac{\sigma \text{ corrected}}{\sigma \text{ uncorrected}} \,.$$

(*) The scattering length α can be defined as

$$\lim_{k \to 0} k \operatorname{cotg} \delta = -\frac{1}{\alpha} = (\varkappa+\mu) \left[\frac{8\pi}{a_0} + l(\varkappa+\mu)^2 - l(s_0) \right] .$$

We have put in the figure the ratio

$$R = \frac{\sigma \text{ corrected}}{\sigma \text{ uncorrected}},$$

in function of the kinetic energy of the π-meson in the laboratory system ($K_L = k\sqrt{s}/\varkappa\mu$). The quantity $|R-1| = \Delta$ measures the correction and we can deduce the following results:

a) Δ is more important in the forward scattering than in the backward scattering.

b) Δ increases with the energy as expected.

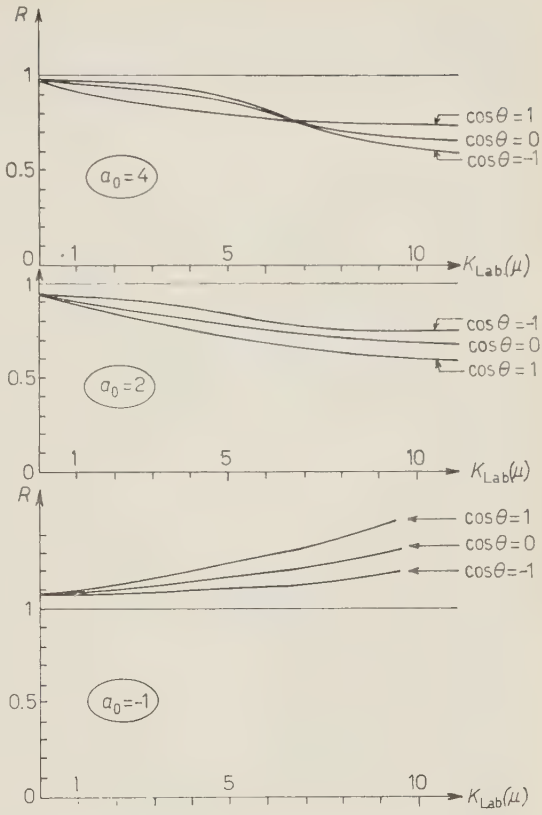


Fig. 8.

In conclusion, the limit of validity of the effective range approximation corrected by the crossing terms seems to be sufficiently below the threshold for the next intermediate state in the unitarity condition [$s=(\varkappa+3\mu)^2$ corresponds to $K_L = 7.3\mu$]. Nevertheless, the iteration procedure appears as rapidly convergent in the low energy model and one can hope to obtain a good solution after another iteration with a larger domain of validity. But for this, more important numerical computations are necessary.

6·6. – We now introduce the left hand cut contributions due to channel III. If they are not very important we can simply add these to the form (46). Under this assumption, the scattering amplitude can be written

$$\frac{1}{8\pi} G^{(I)}(W, \cos\theta) = \left[M(s) - i\frac{k}{W}\right]^{-1} + \frac{1}{M(s_2)} - a_0 + \frac{1}{8\pi} g^{(I)}(t, Y), \tag{47}$$

and the splitting between the two isospin states is a test of the importance of the two-pion contributions.

If the two-pion term is large, one must use more refined methods to solve our integral equations. It can also give the total contribution to π-K scattering if we believe the bipion model as correct. We do not treat this problem here.

7. – Concluding remarks.

We have obtained a type of solution for the π-K scattering depending of two arbitrary parameters that one can interpret as the two S scattering lengths in the isospin states $I = \frac{1}{2}$ and $I = \frac{3}{2}$.

The most important step seems us to be the investigation of a possible resonance in the P-waves. This can be the explanation of some experimental data of π-meson production in K-$\mathcal{N}$ scattering. In appears that such a resonance can be induced by the corresponding one for π-π scattering in the $I = J = 1$. With an opposite sign of the two-pion contribution to the π-K scattering amplitude one can presume that a π-K resonance exists in the $I = \frac{3}{2}$ state and not in the $I = \frac{1}{2}$ state. Preliminary estimations in an effective range approach make possible such a situation.

* * *

We are indebted to Dr. A. MARTIN for illuminating discussions on the problem of solutions for an equation of Low type.

As we were writing down this paper, a preprint by B. W. LEE, on the same subject appeared. We acknowledge it for interesting discussion.

RIASSUNTO (*)

Si è proposta una interazione diretta mesone π-mesone K come uno dei possibili processi responsabili delle interazioni mesone K-nucleone. Si cerca in questo scritto di vedere ciò che in merito a tale interazione permette di determinare la rappresentazione di Mandelstam, nella forma semplificata e realistica data da CINI e FUBINI. Si dà una tecnica generale di calcolo per le diverse onde parziali e si calcola il contributo di una interazione diretta mesone π-mesone π all'ampiezza di scattering mesone π mesone K. Infine, facendo l'ipotesi di un'onda S importante, si dà per quest'ultima una espressione nell'approssimazione della portata effettiva e si valuta l'importanza delle correzioni dovute al possibile incrocio dei mesoni π. Lo studio delle onde P non viene affrontato in questo scritto ed è in corso di studio.

(*) *Traduzione a cura della Redazione.*

Total Neutron Cross-Section of P and S in the 3 to 5 MeV Energy Range (*).

P. Cuzzocrea, S. Notarrigo, R. Ricamo and F. Vinci

Istituto di Fisica dell'Università - Catania
Centro Siciliano di Fisica Nucleare - Catania

(ricevuto il 18 Luglio 1960)

Summary. — The total neutron cross sections of P and S have been measured for neutron energies between about 3 and 5 MeV. Transmission measurements were made in steps of 25 to 30 keV with a neutron energy spread of 30 to 60 keV. Neutrons were produced from the ^{2}D(d, n)^{3}He reaction with 2 MeV deuterons. Neutrons of various energies were obtained at various angles from the direction of the deuteron beam. The average total cross section of the two neighbour nuclei at 4 and 5 MeV suggests the opportunity of a refinement of the optical model.

1. – Introduction.

The optical model (1) with the refinements of a potential well with rounded edges and a spin-orbit interaction term (2) allows to describe the gross features of the total cross-section as a function of the atomic weight for interacting neutrons from 7 to 14 MeV (3).

However for 4 MeV neutrons there is not a good fit of the experimental data with theoretical predictions (3) and there is a lack of experimental data mainly in the $20 < A < 40$ region.

(*) Work sponsored in part by the U.S.A. Air Force and C.R.R.N. (Sicily).

(1) H. Feshbach, C. E. Porter and V. F. Weisskopf: *Phys. Rev.*, **96**, 448 (1954).

(2) W. B. Riesenfeld and K. M. Watson: *Phys. Rev.*, **102**, 1157 (1956).

(3) F. Bjorklund and S. Fernbach: *Phys. Rev.*, **109**, 1295 (1958); A. Bratenahl, J. M. Peterson and J. P. Stoering: *Phys. Rev.*, **110**, 927 (1958).

Besides one can observe some regularities, comparing the behaviour of the total neutron cross-sections, in the region between 3 and 5 MeV for neighbour nuclei (Fig. 1), with the behaviour of the binding energies of the last neutron in the compound nuclei.

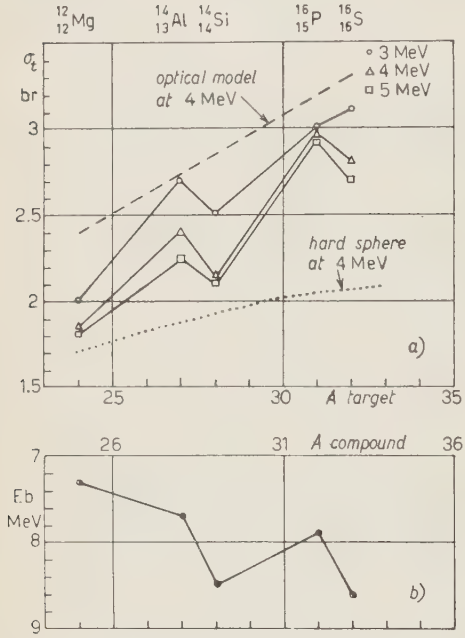


Fig. 1. – *a*) Experimental total cross sections of 3, 4 and 5 MeV neutrons as a function of A (full line) and total cross section for the potential scattering from a hard sphere for 4 MeV neutrons (dotted line), and that calculated with the optical model (dashed line) ([3]); *b*) binding energies of the last neutron in compound nuclei.

This evidence suggests that in a more refined optical model the potential well (depth, range and shape) should depend on A also according to the requirements of the shell model as was pointed out by RAINWATER ([4]).

In view of these facts we have measured the total cross-section as a function of the energy in the $(3 \div 5)$ MeV range for ^{31}P and ^{32}S that are nuclei differing only by one proton and with the last nucleons in the $2s$ state. Moreover for these elements there are not many data; in fact the total neutron cross-sections for phosphorus and sulphur have been measured from 1.90 to 3.67 MeV by RICAMO ([5]) with a mean resolution of about 50 keV and from 3.0 to 13 MeV by NERESON and DARDEN ([6,7]) who have measured the average total neutron cross-section with poor energy resolution (about 10%).

However the absolute values given by NERESON and DARDEN ([6,7]) are not in agreement with those obtained by WEIL and JONES ([8]) for some elements and also for this reason we decided to extend the measurements with good resolution for phosphorus and sulphur up to 5 MeV in our program of systematic study of nuclei with about $A = 30$. This is the region where in the neutron energy range from 3 to 5 MeV the isolated resonances disappear and the cross-

([4]) J. RAINWATER: *Handb. d. Phys.*, **40**, 422 (1957).
([5]) R. RICAMO: *Nuovo Cimento*, **8**, 383 (1951).
([6]) N. NERESON and S. E. DARDEN: *Phys. Rev.*, **94**, 1678 (1954).
([7]) N. NERESON and S. E. DARDEN: *Phys. Rev.*, **89**, 775 (1953).
([8]) J. L. WEIL and K. W. JONES: *Phys. Rev.*, **110**, 466 (1958).

section shows only small variations due probably to fluctuations in the level density.

However recent measurements on the $^{31}P(n, p)^{31}Si$ cross-section made in this laboratory ([9]) show some resonances and it is therefore interesting to investigate if these are also present in the total cross-section.

2. – Experimental method.

2·1. *Neutron source.* – We have used the $^{2}D(d, n)^{3}He$ reaction on a heavy ice target as monoenergetic neutron source. The deuterons produced by ionizing D_2 gas in a RF source were accelerated by a HVEC 2 MeV Van de Graaff electrostatic accelerator to the energy of 2 MeV. The deuteron beam was magnetically deflected by 25° and focused on the ice target.

The heavy ice target has been obtained by condensing a D_2O vapor stream on a copper tube properly shaped and cooled with liquid air. The target thickness was usually about 150 keV for 2 MeV deuterons and the deuteron beam intensity was about 4 μA. The neutron flux remained constant within a few per cent during some hours. The most part of the neutron energy indetermination was due to the target thickness, having our Van de Graaff 2‰ stability this corresponding to only 4 keV indetermination for the neutron energy.

According to FOWLER and BROLLEY tables ([10]) and taking into account the energy loss in the target for the deuterons used ($E_d = 2.0$ MeV), the neutron energy E_n was varied from 2.42 to 5.17 MeV by changing the angle θ from 108° to 0° (usually in steps of one degree). The neutron energy uncertainty due to the heavy ice target thickness was usually of about 70 keV in the forward direction but only 30 keV at 90°.

During the course of measurements we have very often tested the Van de Graaff voltage by observing the resonance peaks of the $(p, \alpha\gamma)$ reaction in ^{19}F at 874 keV ([11]) using a CaF_2 thin evaporated target. The threshold for the $^{7}Li(p, n)^{7}Be$ reaction was also used as standard calibration point at 1.881 MeV ([12]).

2·2. *Neutron detection.* – Neutrons were detected by a Hornyak scintillator *D* (Fig. 2), 25 mm in diameter and 20 mm thick, in order to discriminate against the γ rays from the reaction $^{16}O(d, p\gamma)^{17}O$ ($Q = 1.92$ MeV; $E_\gamma = 0.87$ MeV) due to the presence of ^{16}O in the heavy ice target. The γ-yield of this reaction

([9]) P. CUZZOCREA, G. PAPPALARDO and R. RICAMO: *Nuovo Cimento*, **16**, 450 (1960).
([10]) J. L. FLOWER and J. E. BROLLEY: *Rev. Mod. Phys.*, **28**, 103 (1956).
([11]) S. E. HUNT and K. FIRTH: *Phys. Rev.*, **99**, 786 (1955).
([12]) *Conference of the National Research Council.* Subcommittee of Nuclear Constants (April 29, 1959).

is indeed remarkable for 2 MeV deuterons ([13]), (about 60% of the neutron yield).

The Hornyak scintillator was coupled to a EMI 9524/B photomultiplier tube followed by conventional electronics. The discriminator bias was set high enough to eliminate the γ rays of a standard source of Ra. The detector was at 60 cm from the source.

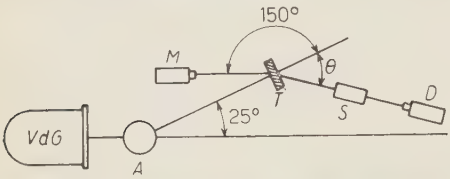


Fig. 2. – Experimental arrangement in a horizontal plane, schematic.

The neutron background from spurious sources has been checked by measuring the transmission for various thick scatterers (Fe, Pb, plexiglass) and it was found to be 3% at 0° and 10% at 90°.

The neutron source intensity was monitored by a second scintillation neutron detector *M* frequently checked with a current integrator on the deuteron beam. The monitor was placed at 150° from the deuteron beam.

2·3. *Samples.* – Several phosphorus samples have been obtained by introducing chemically pure red phosphorus powder in thin iron cylinders. Cylinders were ermetically sealed up at the ends, with thin brass disks tin soldered, in order to exclude air. We have observed, in fact, an increase of the weight of red phosphorus exposed to air amounting to about 50% in some months. Chemical analyses conducted in our laboratory have shown the formation of H_3PO_3 in red phosphorus samples exposed to air but none in phosphorus samples ermetically sealed up (*).

The diameter of the phosphorus samples was about 23 mm and the lengths were around 140 mm in order to have a transmission of about 0.4.

Two samples of sulphur have been made by using pure sulphur crystals melted in cylindrical form, 23 mm in diameter, around 90 mm long to have a transmission of 0.4.

The samples' constants have been measured with a precision better than 1%.

The scatterers *S* and the detector *D* (Fig. 2) were placed on a suitable support mounted on a system which could be rotated arond the vertical centered on the target. The scatterer could be replaced by a plexiglass cylinder 250 mm long and 24 mm in diameter, used by us as scatterer with very low and known transmission coefficient *R*. The sample and the plexiglass scatterer were placed midway between the source and the detector, in order to realize the minimum neutron inscattering ([5]). We have carefully alined the samples but a lateral

([13]) R. G. THOMAS and T. LAURITSEN: *Phys. Rev.*, **88**, 969 (1952).

(*) We thank Dr. S. PRIVITERA who has conducted these analyses.

displacement of about 2 mm did not produce a detectable variation in the transmission.

The true $\theta = 0°$ position of the detector was found measuring carefully the angular distribution of neutrons at symmetrical sides respect to the deuteron beam direction.

2·4. *Treatment of data.* – The total neutron cross-section was measured by means of the transmission technique using the relation:

$$\sigma_t = -(\ln T)/n \,, \tag{1}$$

where n is the number of nuclei per cm² of the sample and $T = I_1/I_0$ is the transmission coefficient, *i.e.* the ratio between the intensity I_1 of neutrons at the detector with the sample interposed between the detector and the neutrons source, and the intensity I_0 at the detector when no sample is interposed. The transmission of the sample, has been measured using the relation

$$T = 1 - (1 - R)\frac{r_0 - r_1}{r_0 - r_2} \,, \tag{2}$$

which assumes ([5]) that the background is proportional to the intensity of direct neutrons and accounts for any instability of the neutron source intensity. r_0, r_1, r_2 are the numbers of neutrons counted by the main detector divided by the monitor counts respectively without the sample, with the sample, and with the known absorber R between the neutron source and the main detector. A plexiglass sample with a known transmission $R = 0.02$ was used as absorber in order to measure the background neutrons at the main detector.

3. – Results.

The total cross-section has been computed by using the relations (1) and (2). As the maximum contribution to the error of T is due to r_1, in each measurement we measured r_1 twice. Our measurement sequence was: r_1, r_0, r_2, r_1.

We have repeated the T measurements seven times in the whole energy range for the two elements and we used the average value to calculate σ_t. The mean square errors of this value are 1.5%.

The inscattering corrections were calculated according to VERVIER ([14]) using the equation

$$\frac{\Delta\sigma_t}{\sigma_t} = \frac{\omega_1\omega_2}{\omega_0}\frac{\sigma(\vartheta)_{0°}}{\sigma_t} \,. \tag{3}$$

([14]) J. S. VERVIER: *Nucl. Instr.*, **2**, 53 (1958).

Our geometrical factor was $\omega_1\omega_2/\omega_0 = 2.5 \cdot 10^{-2}$ and the anisotropy factor $\sigma(\vartheta)_{0°}/\sigma_t = 0.33$. The relative error in σ_t due to the inscattering effect was therefore 0.8%.

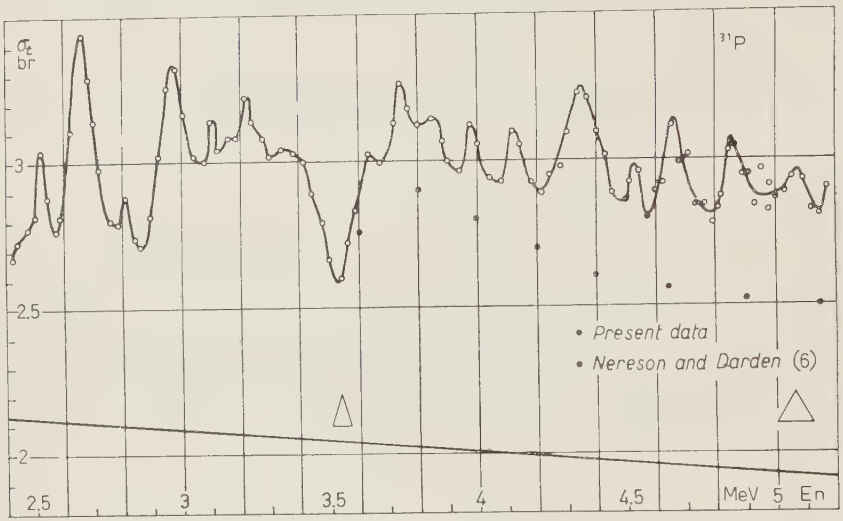


Fig. 3. – Total neutron cross section for ^{31}P, and hard sphere potential scattering ($r_0 = 1.5$ fermi).

The neutron energy was corrected for the heavy ice target thickness.

Our results are shown in Fig. 3 and 4. These results are in good agree-

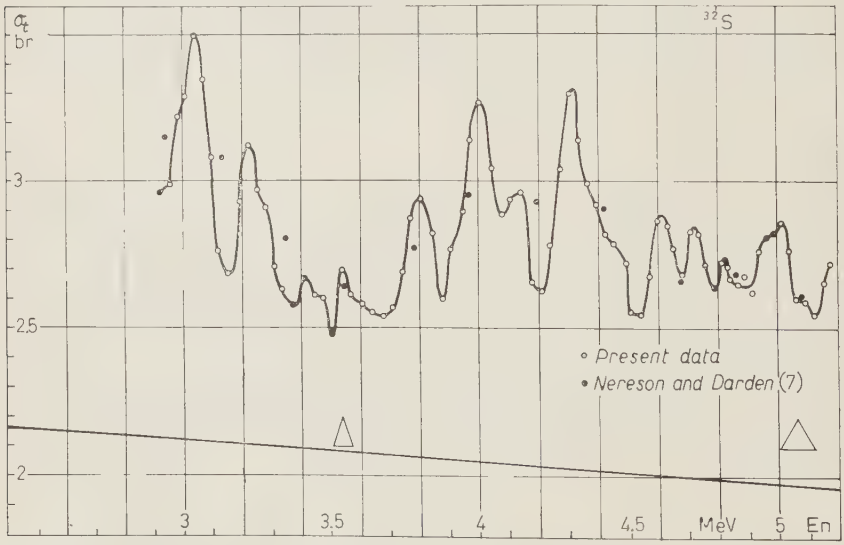


Fig. 4. – Total neutron cross section for ^{32}S, and hard sphere potential scattering ($r_0 = 1.5$ fermi).

ment with those of RICAMO [5] in the range of overlapping energies. For higher energies the σ_t for sulphur is only a little higher than that found by NERESON and DARDEN [7]. The σ_t for phosphorus, for energies from 3.6 to 5.2 MeV is about 15% higher than that given by NERESON and DARDEN [6]. In order to exclude the possibility that there were systematic errors in our measurement, we have tested our whole technique measuring the total neutron cross-section for aluminium and carbon. Our results are practically coincident with other ones [15-19]

4. – Conclusions.

Total neutron cross-sections averaged from our measurements around 4 and 5 MeV for phosphorus and sulphur are in agreement with the behaviour of total cross-sections for the other similar doublet Al–Si at the same energies (Fig. 1). This agreement did not result from previous measurements [6,7] where at 4 and 5 MeV the neutron total cross-section for sulphur had a higher value than that for phosphorus.

Only at 3 MeV there is a disagreement with the above mentioned regularities but we think it should be interesting to improve the optical model in order to explain these regularities.

Namely it would be probably useful to establish the parameters of the potential well taking into account some properties of the shell model for each single nucleus. Besides the comparison of total neutron cross-section of ^{31}P with the ^{31}P(n, p)^{31}Si cross-section [9] shows some correspondences between the peak structure of the two cross-sections expecially at $E_n = 3.72$ MeV.

---

[15] R. MEIER, R. RICAMO, P. SCHERRER and W. ZÜNTI: *Helv. Phys. Acta*, **26**. 451 (1953).

[16] R. RICAMO and W. ZÜNTI: *Helv. Phys. Acta*, **24**, 419 (1951).

[17] K. C. BOKELMANN, D. W. MILLER, R. K. ADAIR and H. H. BARSHALL: *Phys. Rev.*, **84**, 69 (1951).

[18] J. E. WILLIS, J. K. BAYER, H. O. COHN and B. H. WILLARD: *Phys. Rev.*, **109**. 891 (1958).

[19] R. L. BECKER and H. H. BARSHALL: *Phys. Rev.*, **102**, 1384 (1956).

RIASSUNTO

È stata misurata la sezione d'urto totale del fosforo e dello zolfo per neutroni di energia compresa tra 3 e 5 MeV, con risoluzione da 30 a 60 keV ad intervalli di circa 30 keV. La σ_t mediata tra 4 e 5 MeV per il $^{16}_{15}$P risulta maggiore di quella dello zolfo $^{16}_{16}$S, contrariamente a quanto trovato da altri Autori. I risultati sperimentali, posti in relazione all'andamento dell'energia di legame dell'ultimo neutrone nel nucleo composto al variare del numero di massa, suggeriscono l'opportunità di un perfezionamento del modello ottico nel senso di tenere conto delle proprietà del modello a shell.

3901

Deuteron Production in Pick-Up Reactions – I.

E. G. Beltrametti, L. Ciuffolotti and G. Tomasini

Istituto di Fisica dell'Università - Genova
Istituto Nazionale di Fisica Nucleare - Sezione di Genova

(ricevuto il 19 Luglio 1960)

Summary. — Some features of the deuteron production, by 120 MeV protons interacting in nuclear emulsions, are examined, which exhibit deviations from a simple independent particle model of the nucleus. In part I of this work the experimental results and the technique used to discriminate the deuterons are described. In part II we shall be concerned in with the interpretation of the experimental data.

1. – Introduction.

Among the reactions produced by bombarding atomic nuclei with rather high energy nucleons, the (p, d) and (n, d) ones have been successfully used to study the internal nuclear structure [1-11].

The secondary deuterons coming out in these collisions are, as a first ap-

[1] G. F. Chew and M. L. Goldberger: *Phys. Rev.*, **77**, 470 (1950).
[2] W. Selove: *Phys. Rev.*, **101**, 231 (1956).
[3] W. N. Hess and B. J. Moyer: *Phys. Rev.*, **101**, 337 (1956).
[4] J. Hadley and H. York: *Phys. Rev.*, **80**, 345 (1950).
[5] K. Brueckner and W. M. Powell: *Phys. Rev.*, **75**, 1274 (1949).
[6] H. Bradner: *Phys. Rev.*, **75**, 1467 (1949).
[7] J. Heidmann: *Phys. Rev.*, **80**, 171 (1950).
[8] B. H. Bransden: *Proc. Phys. Soc.*, A **65**, 738 (1958).
[9] K. J. Le Couteur: *Proc. Phys. Soc.* A **63**, 259 (1959).
[10] I. Dostrovsky, P. Rabinowits and R. Bivins: *Phys. Rev.*, **111**, 1659 (1958); J. Combe: *Suppl. Nuovo Cimento*, **3**, 182 (1956).
[11] R. W. Deutsch: *Phys. Rev.*, **97**, 1110 (1955).

proximation, the result of two essentially different processes. First, they can be produced by interactions between the incident particle and a small number of target nucleons, during a time of the order of 10^{-22} s required in traversing the nucleus. Second they can be « evaporated », after a relatively long time, by the residual nucleus, usually left in an excited state. The last process contributes to the emission of low energy ($< \sim 20$ MeV) deuterons whose features (*e.g.* energy spectrum and angular distribution) are satisfactorily accounted for by the nuclear evaporation theory ([9,10]).

Coming back to the first process, the usual pick-up mechanism, based on a nuclear independent particle model (hereinafter IPM), describes in a satisfactory way the production of high energy deuterons, *i.e.* of deuterons whose energy is close to that of the incoming nucleon decreased by the binding energy of the picked-up nucleon.

However there is, in general, a wide range of intermediate energies, where neither of the mechanisms we have quoted can give rise to the appreciable deuteron production which seems to be suggested by some experimental results ([4-6]). The problem of the production of intermediate energy deuterons is the main object of the present work.

Our experimental analysis has been made by observing the interactions of 120 MeV protons in heavy nuclei of G-5 emulsions. The deuteron discrimination has been obtained by a recent improvement in the technique of the mass discrimination in nuclear emulsions, which will be reported in the next paragraph.

Our experimental results confirm an appreciable production of intermediate energy deuterons (for our case in a range from 10 MeV to ~ 70 MeV) which, as previously pointed out, cannot be explained by a IPM pick-up process. To be noted that, at the energy of the incoming particle in which we are interested (~ 100 MeV), the wave length associated with its motion is of the same order of magnitude as the mean nucleon spacing in nuclear matter.

The study of the intermediate energy deuterons, is expected to give information on the correlation between the nucleons in the nucleus. In fact we shall show that a reasonable agreement with the experimental data can be obtained using, in a rather simplified way, a two-nucleon, or « quasi-deuteron », nuclear model and a generalized pick-up mechanism (Part II).

2. – Experimental technique.

The experimental analysis has been performed in 600 μm nuclear emulsions Ilford G-5 exposed to a (142 ± 3) MeV proton beam; they were under-developed in order to obtain a good mass discrimination by means of ionization measurements. These ionization measurements have been performed with a

semi-automatic apparatus able to give several parameters simultaneously, as extensively discussed in previous papers ([11-14]).

In order to obtain the most convenient conditions for discriminating the masses of protons and deuterons which stop in the emulsion, we considered the following parameters (*):

a) $l(0)$ = gap length for unit length, if the residual range R of the particle is $\geqslant 5$ mm: in this case $l(0)$ has been measured on a length of 200 μm (equivalent to five cells) in the range $4\text{ mm} \leqslant R \leqslant 5\text{ mm}$;

b) the quantity $x_{0,4}$ obtained from x_0 by discriminating the gaps according to their length r, if the residual range R is < 5 mm: more precisely, for $r < r_0 \equiv 0.4$ μm the gap is not considered, while for $r > r_0$ only the exceeding length $r - r_0$ is considered. The distribution of r is strictly exponential for $r > 0.4$ μm ([13]).

Generally it is more convenient to make use of the whole residual range R; *i.e.* to consider the quantity $x_{0,4}(R)$ instead of $x_{0,4}(t)$. To avoid confusion we shall introduce the symbol $Z(R) \equiv x_{0,4}(R)$.

We tested the method by measuring 250 tracks of the primary 140 MeV protons and 615 star branches: in selecting the tracks we required a dip angle $\varphi < 15°$. The results of the mass measurements (**) show the fairly good efficiency of our method.

In Fig. 1 they are shown for the case $R \geqslant 5$ mm and for a total number of 261 tracks coming out from stars. The « confusion region » seems to be quite negligible.

The data of Fig. 1 are also shown on a probability diagram in Fig. 2-*a*; they turn out to lie quite close to two straight lines, *i.e.* according to gaussian distributions. The standard error $\delta M/M$ in the mass determination can be deduced from the slope of the straight lines and turns out to be about 11%. It is to be noted that such a value is less than the one anticipated on the basis of a reasonable model of the track structure ([13,14]); in fact a measure-

([12]) A. De Marco, R. Sanna and G. Tomasini: *Nuovo Cimento*, **9**, 524 (1958).

([13]) G. Cortini, G. Luzzatto, G. Tomasini and A. Manfredini: *Nuovo Cimento*, **9**, 706 (1958).

([14]) L. Ciuffolotti, G. Luzzatto, G. Tomasini and G. Cortini: *Nuovo Cimento*, **9**, 1110 (1958).

(*) We adopt the notation used in references ([13]) and ([14]): t = cell length, $x_0(t)$ = totall length of the gaps in a cell, $l(0) = t^{-1} \cdot x_0(t)$.

(**) They have been partially communicated to the S.I.F. Congress ([15]) which took place at Palermo.

([15]) L. Ciuffolotti, G. Cortini and G. Tomasini: *Suppl. Nuovo Cimento*, **11**, 457 (1959).

ment of $l(0)$ over a 200 μm length would give $\delta M/M \sim 17\%$ (see ref. [14] Table I). The discrepancy is not very surprising if we bear in mind the approximations made [13] to evaluate the theoretical error $\delta M/M$, which gives indeed an over-evaluation.

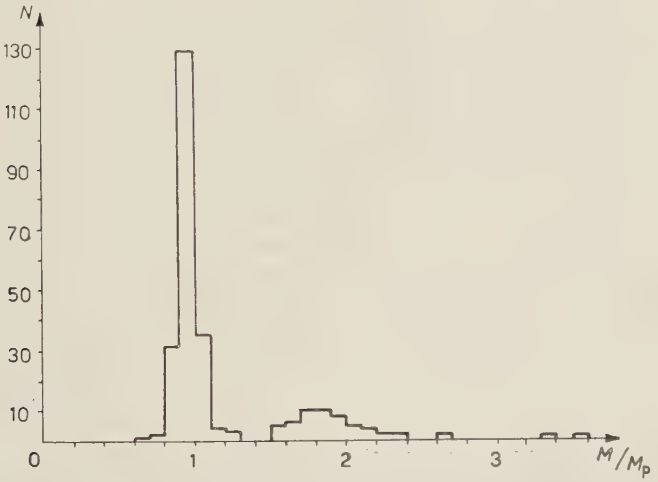


Fig. 1. – Mass distribution obtained measuring $l(0)$ for 261 tracks on a 200 μm length in the range 4 mm $< R <$ 5 mm.

In the case of short tracks ($R < 5$ mm), the dependence of $Z(R)$ on R has to be known in order to obtain a mass determination. The experimental function $Z(R)$ is shown in Fig. 3. In the same figure is shown the function $l(0.4) = t^{-1} \cdot x_{0,4}(t)$ obtained by measuring 20 proton tracks. For comparison we have also reported the function $l(0)$. The curves relative to $l(0.4)$ (deduced by the least squares method) and to $Z(R)$ correspond respectively to the equations

$$l(0.4) = \text{constant} \times R^n ,$$

$$Z(R) = \text{constant} \times R^{n+1} ,$$

where $n = 0.38$.

From this fact it results that there is a simple relation among the expected values of Z relative to tracks of particles of different mass. To be precise, denoting by M_1 and M_2 two different masses and by Z_1 and Z_2 the relative Z values we obtain

$$(M_1/M_2) = (Z_2/Z_1)^{1/n} .$$

The mass distributions of 468 particles, deduced from the measure of $Z(R)$ with $R = 400$ μm are shown in Fig. 4. Also in this case the proton-deuteron

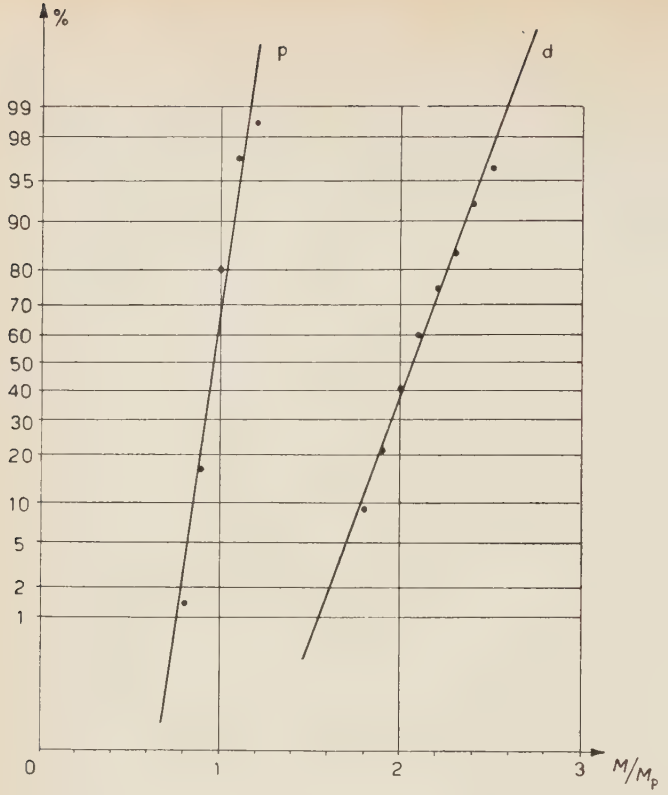


Fig. 2*a*. – Frequency distribution of the masses referred to in Fig. 1, plotted on probability paper.

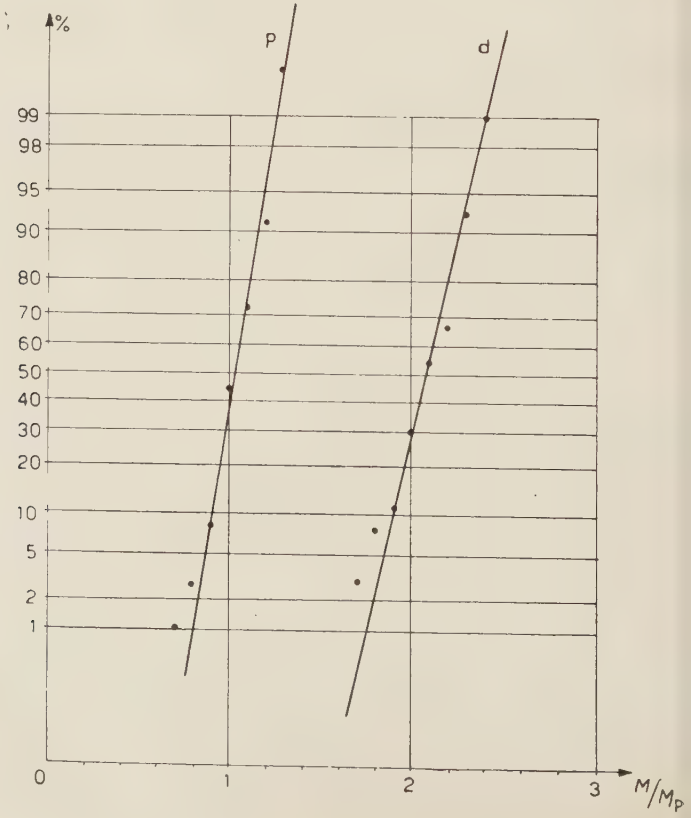


Fig. 2*b*. – Frequency distribution of the masses referred to in Fig. 4, plotted on probability paper.

discrimination is quite satisfactory and the two distributions can be considesed as gaussian ones (see Fig. 2-*b*).

The dispersion of the $Z(R)$ experimental values gives a value of about 12% for the standard error $\delta M/M$. As in the previous case (tracks with $R > 5$ mm)

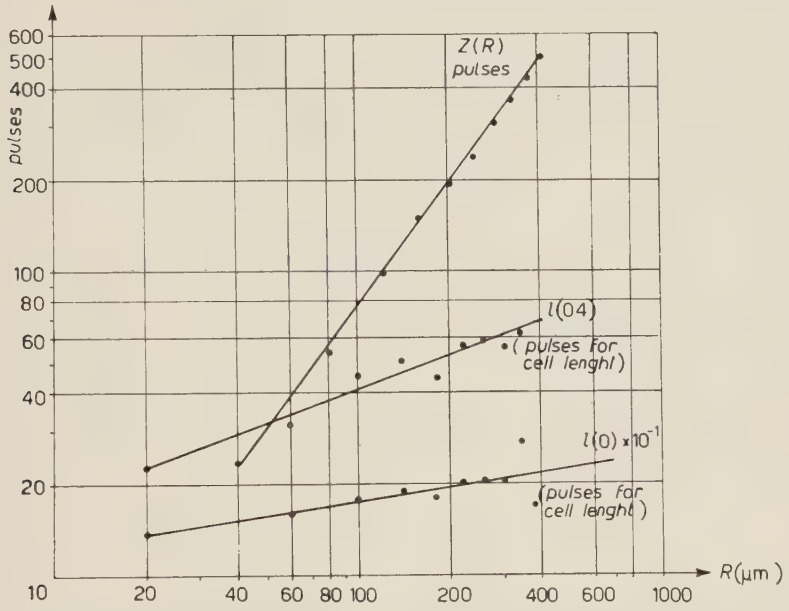


Fig. 3. – Plot of $l(0)=t^{-1}\cdot x_0(t)$, $l(0.4)=t^{-1}\cdot x_{0.4}(t)$, and $Z(R)=x_{0.4}(R)$ against the residual range R.

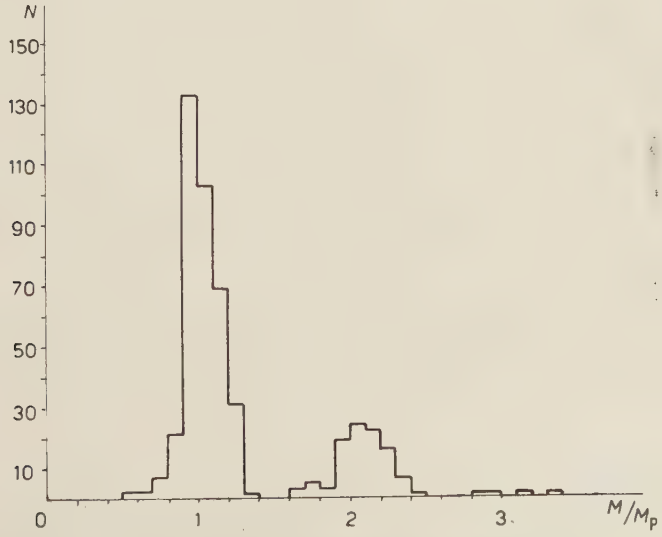


Fig. 4. – Mass distribution obtained measuring Z ($R = 400$ μm) for 468 tracks.

such a figure is less than could be expected from the theoretical evaluation ([13,14]) which would give $\delta M/M \sim 20\%$.

From the data collected in Fig. 1 and in Fig. 4 we can deduce the ratio between the deuteron mass and that one of the proton, which results to be 1.97 ± 0.22 and 2.07 ± 0.24 respectively.

To conclude we can say that the method described is more convenient than those used up to day to discriminate the proton-deuteron masses in nuclear emulsions. So, *e.g.* LANIUS ([16]), by means of scattering measurements on a 300 μm range, cannot give a definite mass assignment to the single track but only a statistical one; TRET'IAKOVA ([17]) obtains an unambiguous mass discrimination only by combining ionization and scattering measurements for long residual ranges.

The time required, with our method, in discriminating the mass of a particle, is about 20 minutes for the case of short tracks (to be measured with method *b*)) while 10 minutes are enough for long tracks (to be measured with method *a*)). We hope that a good mass discrimination can still be obtained for tracks shorter than 400 μm, but it must be noted that the emulsion underdevelopment degree is, in such a case, a very critical parameter (see *e.g.* Table I of ref. ([15])).

3. – Experimental results.

The reactions produced by the incident protons have been found by area scanning and by following the primary tracks. The investigated interactions correspond to an energy of the incoming protons of about 120 MeV.

In the area scanning we have found 3770 events, classified as follows: 572 light nuclei interactions, 77 elastic scatterings, 3121 heavy nuclei (Ag and Br) interactions. The light or heavy nuclei events have been distinguished by means of the usual α-particle method ([18]).

Scanning along the primary tracks, 375 events have been found, classified as follows: 16 light nuclei interactions, 257 elastic scatterings, 102 heavy nuclei interactions (and 3 stoppings). The total track length observed was (2807 ± 10) cm.

Obviously in the area scanning a very low efficiency ($\sim 8\%$) has been found for elastic scattering events and one-prong stars (in which the ionization change is detectable). Such a situation makes our statistics at high energies very poor.

([16]) R. LANIUS: *Nucl. Phys.*, **3**, 391 (1957).

([17]) M. I. TRET'IAKOVA: *Sov. Phys. Journ. Exp. Theor. Phys.*, **5**, 1045 (1957).

([18]) M. G. K. MENON, H. MUIRHEAD and O. ROCHAT: *Phil. Mag.*, **41**, 583 (1950); P. E. HODGSON: *Phil. Mag.*, **45**, 190 (1954).

We have considered only the interactions due to heavy nuclei and studied all branches having a residual range $R \geqslant 400$ μm, a dip angle (with respect to the emulsion plane) $\varphi < 15°$ and stopping in the emulsion. We have measured 861 secondary tracks of which 685 were protons, 164 deuterons and 12 tritons.

The energy spectra of the protons and deuterons are shown in Fig. 5.

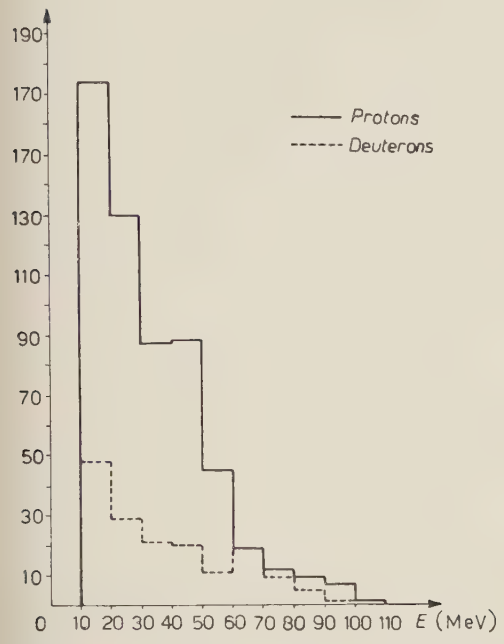


Fig. 5. – Energy spectrum: proton and deuteron number for 10 MeV energy range against the energy.

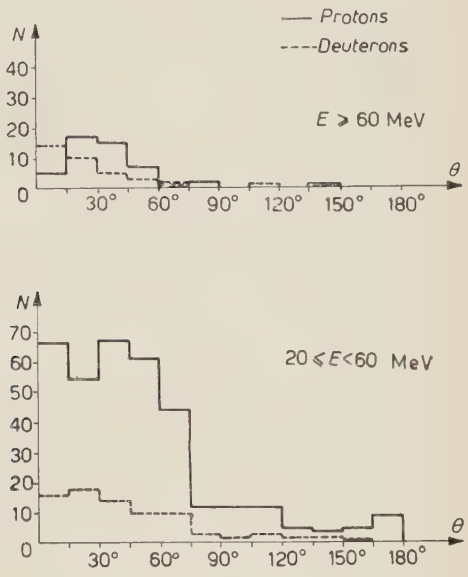


Fig. 6 shows the angular distributions subdivided in three energy ranges (the reason for such a subdivision will become clear afterwards).

Our results, compared with those obtained by other authors [1-4] using different techniques, are in apparent disagreement. Indeed we do not find the deuteron peak at high energies (in our case at ~ 90 MeV) and at small angles, expected on the basis of an IPM pick-up process. The disagreement is however more apparent than real; in fact it is to be attributed to the loss of high energy events which appear almost always in one-prong stars, whose detection efficiency is, as previously noted, quite small. In Fig. 7 the energy

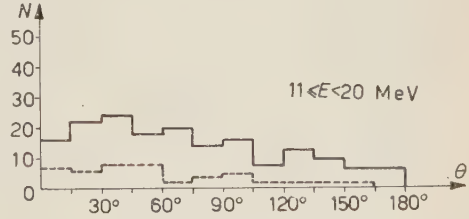


Fig. 6. – Angular distribution for various energy ranges: proton and deuteron number for 15° angle range against the production angle.

spectrum of the deuterons is shown for various types of stars, *i.e.* for stars with 1, 2, 3 or more branches. A high energy deuteron peak can be excluded in the events with two or more branches (as expected) but not in the case of single branch events.

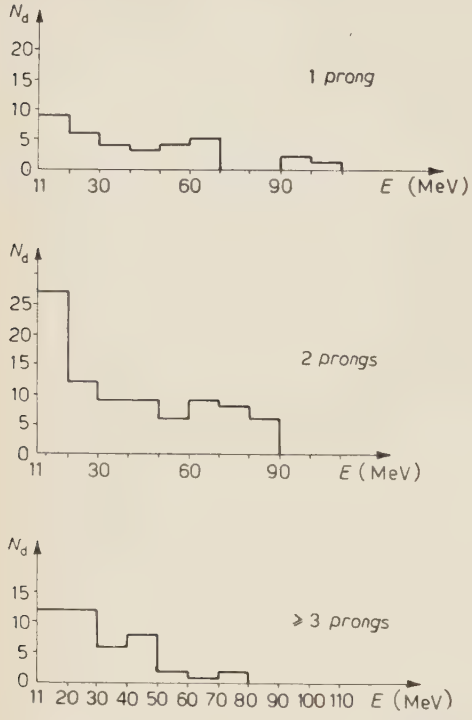


Fig. 7. – Energy spectrum of the deuterons relative to stars having 1, 2, 3 or more prongs.

To conclude we can say that our data are significant for energies less than about 70 MeV.

The interaction cross-section can be deduced from the mean free path of the incident protons and turns out to be, for heavy nuclei (Ag and Br) interactions,

$$\sigma_H = (1.84 \pm 0.2) \text{ barns} .$$

Evaluating the cross-section σ_d for deuteron production, the experimental data are to be corrected taking into account the limitations in the investigated solid angle (complete symmetry around the direction of the incident proton is obviously assumed). Doing so the results quoted in Table I have been obtained.

TABLE I.

Energy range	Proton number		Deuteron number		Deuterons to protons percentage
	exper.	calc.	exper.	calc.	
11 MeV $< E <$ 20 MeV	174	757	48	199	26.4
20 MeV $< E <$ 60 MeV	350	1314	81	293	22.4

Bearing in mind the value of σ_H, for the cross-section σ_d we obtain the following results:

$\sigma_d = (118 \pm 20)$ mb for deuteron production in the energy range $11 \text{ MeV} < E_d < 20 \text{ MeV}$;

$\sigma_d = (170 \pm 20)$ mb for deuteron production in the energy range $20 \text{ MeV} < E_d < 60 \text{ MeV}$.

4. – Conclusions.

The most significant feature of our experimental data is the relatively large production of intermediate energy deuterons (say 20 MeV $< E_d <$ 60 MeV). As already noted in Section **1**, both the nuclear evaporation process and the IPM pick-up process cannot explain the magnitude of the experimental cross section nor its dependence on the energy (Fig. 5) and on the angle (in respect to the incident proton direction; see Fig. 6). So it seems to be reasonable to make use of some pick-up process involving more than a single nucleon of the target nucleus. Some approaches in this direction have been made by HEIDMANN [7] and BRANSDEN [8] but their results are quite different one from the other and neither agrees with our experimental data (*).

This situation requires a theoretical analysis which will be done in the second part of this work.

Considering the very low deuteron energies (11 MeV $< E_d <$ 20 MeV) it could be observed that the almost isotropic angular distribution (Fig. 6) suggests that the nuclear evaporation process is operative. However, it is likely that the process responsible for the production of intermediate energy deuterons also contributes for energies $E_d <$ 20 MeV. This could perhaps explain why the experimental percentage of deuterons relative to protons is larger (see also ref. [16]) than the theoretical percentage predicted by the nuclear evaporation theory [9,10].

* * *

We are indebted to Professor G. CORTINI for suggesting the present work and for many helfpul discussions.

Our thanks are also due to Professor E. PANCINI for a number of useful conversations.

(*) Because of the nuclear model used (Fermi gas) the results of Heidmann and Bransden refer only to energies $E_d > 40$ MeV.

RIASSUNTO

Esaminiamo alcune caratteristiche della produzione di deutoni, ottenuti da interazioni di protoni di 120 MeV in emulsioni nucleari, le quali mostrano deviazioni da un semplice modello nucleare a particelle indipendenti. Nella parte I del lavoro sono descritti i risultati sperimentali e la tecnica usata per la discriminazione dei deutoni. Nella parte II ci occuperemo dell'interpretazione dei dati sperimentali.

Deuteron Production in Pick-up Reactions – II.

E. G. Beltrametti and G. Tomasini

Istituto di Fisica dell'Università - Genova
Istituto Nazionale di Fisica Nucleare - Sezione di Genova

(ricevuto il 19 Luglio 1960)

Summary. — We present an interpretation of the production of deuterons whose energy spectrum and angular distribution cannot be explained by a IPM pick-up process. Use of the two-nucleon, or quasi-deuteron, nuclear model is made, and a reasonable agreement with the experimental data is obtained.

1. – Introduction.

The deuteron pick-up process is usually described as follows [1]: a fast incident nucleon, say a proton, in passing through the nucleus, strikes a neutron with such a momentum that the relative momentum of the proton and neutron can be accomodated in the deuteron wave function. When this occurs it is possible for the proton to pick-up the neutron and emerge as a deuteron whose energy lies in a rather narow range, since all the available energy is taken by the deuteron itself.

It has been already noticed that the momentum distribution of the picked-up nucleon, required to account for the experimental cross-section, has momentum components too large to be found in a nuclear independent particle model (hereafter IPM) [2].

In order to explain the ejection of intermediate energy deuterons (in the sense discussed in Part I) we make the assumption that this high momentum is the result of strong direct interactions between the nucleons. Part of the available energy will then be given to another nucleon, usually escaping from the nucleus.

---

[1] G. F. Chew and M. L. Goldberger: *Phys. Rev.*, **77**, 470 (1950).
[2] K. A. Brueckner, R. J. Eden and N. C. Francis: *Phys. Rev.*, **98**, 1445 (1955).

We shall try to analyse our experimental data using the two-nucleon, or quasi-deuteron, model in which the direct interactions between nucleons are taken into account in a very simplified manner.

In this model the nuclear wave function is taken as the product of a term describing two interacting nucleons, (a « quasi-deuteron » if they are a neutron-proton pair), multiplied by the wave function of the remaining $A - 2$ nucleons. The latter does not appear explicitly in the formalism, being supposed that only the two interacting nucleons participate to the reaction under discussion.

The more convincing use of the quasi-deuteron model has been done in the field of high energy nuclear photodisintegration ([3]), but it was used also for pick-up processes, essentially by HEIDMANN ([4]) and by BRANSDEN ([5]).

As previously pointed out (Part I, Section **4**) the results of these authors do not fit our experimental data; however we think that this fact is not due to the uselessness of the two-nucleon model but rather to the final state anti-symmetrization used by them.

The theoretical analysis, which is going to be done by us, is a very simplified one. In particular we shall not include the effects of the interactions between the incident nucleon and the initial nucleus (beyond the pick-up process) nor the interactions between the produced deuteron and the residual nucleus. So, *e.g.* the cross-section reduction due to the nuclear opacity ([6]), and the smoothing out of the deuteron angular distribution due to scattering processes of the incident nucleon and of the deuteron itself ([*]) will not be calculated.

2. – Use of the two-nucleon model.

To make easier the comparisons we shall follow, as far as possible, the formalism used by HEIDMANN and BRANSDEN. The index zero will be relative to the incoming proton while the indices 1 and 2 will be used for the two interacting nucleons in the initial nucleus. In the present case we are concerned with neutron-proton (quasi-deuteron) and neutron-neutron pairs; the index 2 will always refer to a neutron. In the usual approximation of the two-nucleon

---

([3]) See, also for a more complete bibliography: D. H. WILKINSON: *Ann. Rev. Nucl. Sci.*, **9**, 1 (1959).

([4]) J. HEIDMANN: *Phys. Rev.*, **80**, 171 (1950).

([5]) B. H. BRANSDEN: *Proc. Phys. Soc.*, A **65**, 738 (1952).

([6]) See *e.g.* ref. ([2,3]) of part I.

([*]) A recent theoretical analysis by GREIDER ([7]) shows that this effect is not negligible and can be described well by an optical model for the nucleus.

([7]) K. R. GREIDER: *Phys. Rev.*, **114**, 786 (1958).

model, the wave function of the initial state can be written

$$\psi_i^{(p)} = \Phi^{(p)}(0; 1, 2)\Psi^{(p)}(3, 4, ..., A) \qquad \text{if 1 is a proton,} \tag{1}$$

$$\psi_i^{(n)} = \Phi^{(n)}(0; 1, 2)\Psi^{(n)}(3, 4, ..., A) \qquad \text{if 1 is a neutron,} \tag{1'}$$

where $\Psi^{(p)}(3, 4, ..., A)$, (or $\Psi^{(n)}(3, 4, ..., A)$), represents the state of the $A-2$ nucleons which do not participate in the reaction, while $\Phi^{(p)}(0; 1, 2)$, $(\Phi^{(n)}(0; 1, 2))$, describes the incident proton (index 0) and the nuclear subunit constituted by proton 1, (neutron 1), and neutron 2. The explicit form of $\Phi^{(p)}(0; 1, 2)$ and of $\Phi^{(n)}(0; 1, 2)$ will be given in Appendix A.

According to eqs. (1), we shall ignore the anti-symmetrization between the nucleon pair and the remaining $A-2$ nucleons ([2,4,8]). However we cannot ignore the requirements of anti-symmetry between nucleons 0, 1, 2. The final state wave function can therefore be written

$$\psi_f^{(p)} = \frac{1}{\sqrt{2}}[\Theta^{(p)}(0; 1, 2) - \Theta^{(p)}(1; 0, 2)]\Psi^{(p)}(3, 4, ..., A) \qquad \text{if 1 is a proton,} \tag{2}$$

and

$$\psi_f^{(n)} = \frac{1}{\sqrt{2}}[\Theta^{(n)}(2; 0, 1) - \Theta^{(n)}(1; 0, 2)]\Psi^{(n)}(3, 4, ..., A) \qquad \text{if 1 is a neutron,} \tag{2'}$$

where the square bracketed quantities are antisymmetric wave functions (in respect to the indices 1, 0 and 1, 2 respectively) describing the outgoing deuteron and the residual proton or neutron.

Precisely: $\Theta^{(p)}(0; 1, 2)$ describes a deuteron made up by nucleons 1 and 2 and the recoiling proton 0; $\Theta^{(p)}(1; 0, 2)$ describes the state in which the incident proton is associated with nucleon 2 to form the deuteron, while the recoil is absorbed by proton 1; finally, in the state described by $\Theta^{(n)}(2; 0, 1)$, $(\Theta^{(n)}(1; 0, 2))$, the deuteron is formed by 0 and 1 (by 0 and 2), while the energy and momentum conservation is guaranteed by nucleon 2, (1). The explicit form of the $\Theta^{(p)}$ and $\Theta^{(n)}$ functions will be given in Appendix A.

It can be observed that the use of the anti-symmetric wave function (2') is wholly equivalent to the use of the term $\Theta^{(n)}(2; 0, 1)$ or $\Theta^{(n)}(1; 0, 2)$, since the two identical particles play a symmetric role in the initial state, but the same cannot be said about the wave function (2). Consequently we expect that if nucleons 1 and 2 are a neutron-proton pair (a quasi-deuteron) the anti-symmetrization effects can be important. It is a rather curious fact that a wave function of type (2) has never been used in the literature but the sepa-

---

([8]) J. S. LEVINGER: *Phys. Rev.*, **84**, 43 (1951).

rate terms $\Theta^{(p)}(1;0,2)$ and $\Theta^{(p)}(0;1,2)$ have, as for example in reference (4) and (5) rexpectively. This fact is even more surprising as the results of these two calculations are quite different (*).

3. – Matrix element calculation.

The interaction potential between the incident proton and nucleons 1, 2 will be written in the form, with obvious notations,

$$V = V_{0,1} + V_{0,2} \tag{3}$$

and will be assumed charge independent.

To obtain the transition probability we could either calculate the matrix element of the potential V (occurring for a true three-body collision) or sum up the transition probabilities for $V_{0,1}$ and $V_{0,2}$ (occurring for subsequent two-body collisions). Since the wave length of the incident nucleon is of the same order of magnitude as the average nucleon spacing in a nucleus, the coherent expression is more reasonable.

The transition probability can therefore be written, in first Born approximation (**),

$$w = 2\pi \boldsymbol{S} |V_{if}|^2 \varrho_f , \tag{4}$$

where V_{if} is the matrix element of the interaction potential calculated between the initial state (eq. (1)) and the final state (eq. (2) or (2′)); ϱ_f is the density of final states and $\boldsymbol{S}$ stands for the sum or the average over all unobserved quantities. In calculating the matrix element the integration over the co-ordinates of nucleons 3, 4, ..., A gives unity, so that

$$V_{if}^{(p)} = \langle \psi_f^{(p)} | V | \psi_i^{(p)} \rangle = \frac{1}{\sqrt{2}} \langle \Theta^{(p)}(0;1,2) - \Theta^{(p)}(1;0,2) | V | \Phi^{(p)}(0;1,2) \rangle , \tag{5}$$

and

$$V_{if}^{(n)} = \langle \psi_f^{(n)} | V | \psi_i^{(n)} \rangle = \frac{1}{\sqrt{2}} \langle \Theta^{(n)}(2;0,1) - \Theta^{(n)}(1;0,2) | V | \Phi^{(n)}(0;1,2) \rangle . \tag{5′}$$

(*) The production of a deuteron formed by nucleons 1 and 2 (corresponding to $\Theta^{(p)}(0;1,2)$ is often referred to as « indirect pick-up process », to indicate that it could be regarded as a second order process in which the incident nucleon strikes the nucleon 1, (2), which picks up nucleon 2, (1), to emerge as a deuteron. On the other hand, when the deuteron is made by 0 and 2 (corresponding to $\Theta^{(p)}(1;0,2)$ the process is referred to as direct pick-up (9).

To avoid misunderstandings, we shall not use these conventions. In any case $\Theta^{(p)}(0;1,2)$ and $\Theta^{(p)}(1;0,2)$ obviously describe processes of the same order.

(9) A. G. Sitenko: *Sov. Phys. Usp.*, **2** (67), 195 (1959).

(**) We use $\hbar = c = 1$ units.

On physical grounds the terms

$$\langle \Theta^{(p)}(1;0,2)\,|\,V_{0,1}\,|\,\Phi^{(p)}(0;1,2)\rangle; \qquad \langle \Theta^{(n)}(1;0,2)\,|\,V_{0,1}\,|\,\Phi^{(n)}(0;1,2)\rangle;$$

$$\langle \Theta^{(n)}(2;0,1)\,|\,V_{0,2}\,|\,\Phi^{(n)}(0;1,2)\rangle,$$

are expected to be small compared to the other ones. A rough computation shows that their contribution to the quantities (5) or (5′) is of the order of 5%. Therefore we shall use the approximate expressions

$$(6) \qquad V_{if}^{(p)} = \frac{1}{\sqrt{2}}\Big\{\langle \Theta^{(p)}(0;1,2)\,|\,V_{0,1}+V_{0,2}\,|\,\Phi^{(p)}(0;1,2)\rangle - \\ - \langle \Theta^{(p)}(1;0,2)\,|\,V_{0,2}\,|\,\Phi^{(p)}(0;1,2)\rangle\Big\},$$

$$(6') \qquad V_{if}^{(n)} = \frac{1}{\sqrt{2}}\Big\{\langle \Theta^{(n)}(2;0,1)\,|\,V_{0,1}\,|\,\Phi^{(n)}(0;1,2)\rangle - \\ - \langle \Theta^{(n)}(1;0,2)\,|\,V_{0,2}\,|\,\Phi^{(n)}(0;1,2)\rangle\Big\}.$$

Since in the present analysis we are not concerned in polarization effects, the use of a spin-independent potential seems to be a reasonable approximation. Following reference (4) and (5) we shall use a Yukawa potential (see Appendix A).

4. – Cross-section formula.

To perform the explicit calculation of the matrix elements, some other specifications about the nuclear model adopted are obviously needed; in particular we need the wave function of the quasi-deuteron which, as usual, will be assumed as an approximated (Hulthén) s function for a Yukawa potential (10). The detailed calculation is presented in Appendix A, while in Appendix B we discuss the averaging process over the unobserved quantities (see eq. (4)).

The differential cross-section as a function of the deuteron energy E_d and the angle θ between the momentum $\boldsymbol{k}_0$ of the incident nucleon and the deuteron momentum $\boldsymbol{K}$ (the direction of $\boldsymbol{k}_0$ is obviously a symmetry axis) may be written

$$(7) \qquad \sigma(E_d,\theta)\,\mathrm{d}E_d\,\mathrm{d}\Omega = \frac{C(A)}{A}\,\frac{3}{2\pi^4\varepsilon^3 r_0^3 m}\,\frac{K}{k_0}\,\Gamma(k_0,E_d,\theta)\,\mathrm{d}E_d\,\mathrm{d}\Omega,$$

(10) Also for a more complete bibliography, see: L. C. Gomes, J. D. Walecka and V. F. Weisskopf: *Ann. Phys.*, **3**, 241 (1958); K. Gottfried: *Nucl. Phys.*, **5**, 557 (1958); T. Tagami: *Progr. Theor. Phys.*, **21**, 533 (1959).

where m is the nucleon mass; ε is the mean kinetic energy of the nucleons inside the nucleus; $r_0 A^{\frac{1}{3}}$ is the nuclear radius; $C(A)$ is a factor, depending on the mass number A, which represents the effective number of interacting nucleon pairs (quasi-deuterons) in the initial nucleus; $\Gamma(k_0, E_d, \theta)$ is a function defined in Appendix B (*). The computation of the Γ function cannot be done analytically but requires a double numerical integration.

5. – Results and comparison with the experimental data.

Due to the length of the numerical work (**), up to now we have not studied the dependence of the cross-section upon the incident nucleon energy. We limited ourselves to analyse the energy spectrum and the angular distributions of the ejected deuteron for $k_0^2/2m = 120$ MeV (corresponding to our experimental situation).

The results show that: i) for large angles (say $\theta > 30°$) the contribution to the cross section of the term $\Theta^{(p)}(0; 1, 2)$ in eq. (2) (that one used by BRANSDEN) is the most important; ii) for smaller angles the contribution of the term $\Theta^{(p)}(0; 1, 2)$ predominates at low energies (say $E_D < 40$ MeV), while the terms $\Theta^{(p)}(1; 0, 2)$, $\Theta^{(n)}(2; 0, 1)$ and $\Theta^{(n)}(1; 0, 2)$, (*i.e.* the mechanism proposed by HEIDMANN), are effective at higher energies; iii) the contribution of the interference terms is negligible.

The agreement with the experimental data can be considered as satisfactory.

In Fig. 1 the deuteron energy spectrum integrated over θ is shown. It is not surprising that in the range from 10 MeV to 20 MeV the observed deuteron number is significantly larger than the theoretical value. In fact at these

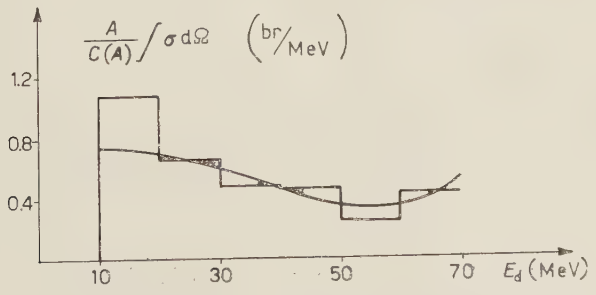


Fig. 1. - Calculated energy spectrum. Comparison is made with the experimental spectrum (see Fig. 5, part I).

(*) See Appendix B, also for a more complete definition of the symbols.

(**) Work which has been done with the IBM 650 computer by Drs. CHIARINI, GIAMBUZZI and LOLLI (Bologna University) to whom we express our deep gratitude,

low energies the nuclear evaporation process is certainly operative. From inspection of Fig. 1 the percentage of evaporated deuterons in such an energy range can be evaluated to about 35 %.

In Fig. 2 the theoretical and experimental angular distributions are compared for the energy range 20 MeV $< E_d <$ 60 MeV.

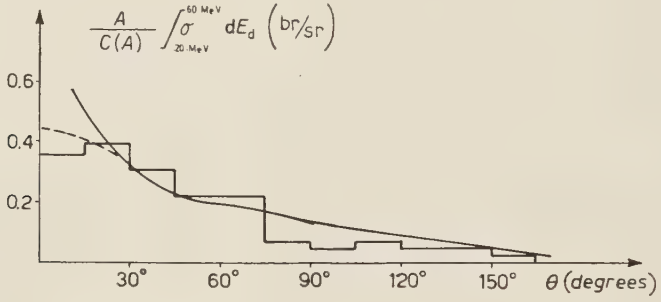


Fig. 2. – Calculated angular distribution (for 20 MeV $< E_d <$ 60 MeV) compared with the experimental one (see Fig. 6, part I). The dotted curve shows the energy dependence of the cross section for a fixed energy (E_d=40 MeV). The two curves practically coincide for $\theta > 30°$.

The rapid increase in the calculated angular distribution for mall angles comes from the upper part of the energy spectrum. Such a feature cannot be unambiguously tested, owing to the poor statistics.

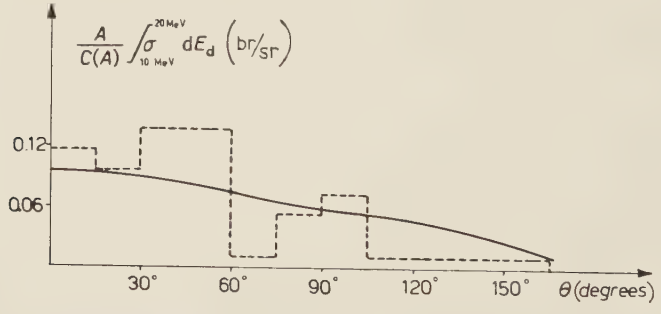


Fig. 3. – Angular distribution in the range 10 MeV $< E_d <$ 20 MeV. The dotted line shows the experimental results corrected for the contribution of the evaporated deuterons (see Fig. 6, part I).

The calculated angular distribution for the range 10 MeV $< E_d <$ 20 MeV is shown in Fig. 3. According to the interpretation of Fig. 1 previously given, a comparison with the experimental data should be done by subtracting from the curve reported in Fig. 6, Part I, an isotropic distribution equivalent to

about 35% of the deuterons observed in this energy range. However, due to the poor statistics, the comparison is not very significant.

Finally, to obtain a quantitative agreement between the experimental cross-section (see Part I, Section **3**) and the calculated one, we must take

$$\frac{C(A)}{A} \sim 0.9\% .$$

Thus, in our case, the effective number of interacting nucleon pairs turns out to be about unity.

It is to be remembered that our definition of $C(A)$ includes the nuclear absorption effect on the incident proton and on the produced deoteron. The last is thought to be quite important but unfortunately it is difficult to make an unambiguous evaluation of it.

* * *

The authors are indebted to Professors A. Borsellino, G. Cortini and A. Gamba for a number of useful discussions.

Appendix A

Matrix element calculation.

We shall adopt the laboratory frame of reference in which the C.M. of the initial nucleus is at rest; in the quasi deuteron model approximation and for rather heavy nuclei it will coincide with the C.M. system of the residual nucleus.

Let us denote by $\boldsymbol{r}_i$ $(i = 0, 1, 2)$ the position vector of the i-th nucleon, by $\boldsymbol{k}_i$ its initial momentum and by $\boldsymbol{K}$ the momentum of the produced deuteron (*). The following notation will also be useful:

$$\boldsymbol{r} = \tfrac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2); \qquad \boldsymbol{\rho} = \boldsymbol{r}_1 - \boldsymbol{r}_2;$$
$$\boldsymbol{Q} = \tfrac{1}{2}(\boldsymbol{k}_1 + \boldsymbol{k}_2); \qquad \boldsymbol{q} = \tfrac{1}{2}(\boldsymbol{k}_1 - \boldsymbol{k}_2) .$$

The wave function for the initial state can be written in the form

$$\Phi^{(\lambda)}(0; 1, 2) = \exp\left[i\boldsymbol{k}_0 \cdot \boldsymbol{r}_0\right] s_0 \frac{1}{\sqrt{v}} \exp\left[2i\boldsymbol{Q} \cdot \boldsymbol{r}\right] \varphi_q^{(\lambda)}(\boldsymbol{\rho}) s_{1,2}^{(\lambda)} , \tag{A.1}$$

(*) We shall use $\hbar = c = 1$ units.

where $\lambda = p$, $(\lambda = n)$, if particle 1 is a proton, (neutron). s_0, $s_{1,2}^{(p)}$, $s_{1,2}^{(n)}$ are the spin functions of the incoming proton, of the proton-neutron and of neutron-neutron pair respectively; $\varphi_q^{(\lambda)}(\boldsymbol{\varrho})$ describes the relative motion of nucleons 1 and 2; v is the normalization volume for the plane wave which describes the C.M. motion of the nucleon pair (quasi-deuteron) and will be taken as the nuclear volume; the plane wave of the incident proton will be normalized in a unit volume.

For $\varphi_q^{(\lambda)}(\boldsymbol{\varrho})$ we shall take, as usually, a Hulthén approximated s function for a Yukawa potential

$$\varphi_q^{(\lambda)}(\boldsymbol{\varrho}) = c^{(\lambda)}\, \varrho^{-1} \left[\frac{\sin(q\varrho + \delta^{(\lambda)})}{\sin \delta^{(\lambda)}} - \exp[-\mu^{(\lambda)}\varrho] \right]. \tag{A.2}$$

$(\mu^{(\lambda)})^{-1}$ is the range of the Yukawa potential and $\delta^{(\lambda)}$ is the phase shift of the s wave, which, from the « effective range » theory, is approximatively given by

$$\operatorname{ctg} \delta^{(\lambda)} \sim -\frac{a^{(\lambda)}}{q}, \tag{A.3}$$

$(\alpha^{(\lambda)})^{-1}$ being the scattering length (for the proton-neutron scattering when $\lambda = p$, or neutron-neutron scattering when $\lambda = n$). The normalization constant $c^{(\lambda)}$ is usually found by requiring the plane wave, of which eq. (A.2) is the s term when $\varrho \gg (\mu^{(\lambda)})^{-1}$ and $q\varrho \gg \delta^{(\lambda)}$, to be normalized in a nuclear volume v around the neutron 2 ([4,8]). We find, in this way,

$$\varphi_q^{(\lambda)}(\varrho) = \frac{\varrho^{-1}}{\sqrt{v}\sqrt{\alpha^{(\lambda)2} + q^2}} \left[-\frac{\alpha^{(\lambda)}}{q} \sin q\varrho + \cos q\varrho - \exp[-\mu^{(\lambda)}\varrho] \right]. \tag{A.2'}$$

The spin function $s_{1,2}^{(p)}$, (eq. (A.1)), is an incoherent superposition, with equal weights, of the four possible spin states for the nucleon pair, while $s_{1,2}^{(n)}$ must be the singlet function, as required by the exclusion principle.

The wave functions of the final state can be written

$$\Theta^{(\lambda)}(i; l, m) = \exp[i\boldsymbol{k}' \cdot \boldsymbol{r}_i]\sigma_i \exp[i\boldsymbol{K}(\boldsymbol{r}_l + \boldsymbol{r}_m)/2]\, \varphi_d(\boldsymbol{r}_l - \boldsymbol{r}_m)\sigma_{l,m}, \tag{A.4}$$

where $(i; l, m)$ is any permutation of the indices 0, 1, 2 which appears in eq. (2); $\boldsymbol{k}'$ is the momentum of the i-th nucleon and σ_i its spin function; φ_d describes the internal motion of nucleons l, m and $\sigma_{l,m}$ is the spin part of their wave function. It is to be noted that eq. (A.4) depends upon the nature of nucleon 1 only through the possible permutations $(i; l, m)$ relative to the two values of λ (see eq. (2)).

For φ_d we shall take the usual Hulthén approximate wave function

$$\varphi_d(|\boldsymbol{r}_l - \boldsymbol{r}_m|) = A_d(\exp[-\alpha_1|\boldsymbol{r}_l - \boldsymbol{r}_m|] - \exp[-\alpha_2|\boldsymbol{r}_l - \boldsymbol{r}_m|])\,|\boldsymbol{r}_l - \boldsymbol{r}_m|^{-1}, \tag{A.5}$$

and $\sigma_{l,m}$ will describe the triplet spin state.

To simplify the numerical computations (which are quite laborious) and as a first approximation, we shall assume that the interaction potential between

the incident nucleon and nucleons 1, 2 (see eq. (3)), is a spin-independent central potential with a Yukawa shape

$$V_{0,i} = U \frac{\exp[-\mu|\boldsymbol{r}_0 - \boldsymbol{r}_i|]}{\mu|\boldsymbol{r}_0 - \boldsymbol{r}_i|}, \qquad (i = 1, 2). \tag{A.6}$$

Accordingly, we shall neglect the dependence of eq. (A.2), or eq. (A.2′), on the nature of nucleon 1. In other words we shall take $\alpha^{(p)} = \alpha^{(n)} = \alpha$ and $\mu^{(p)} = \mu^{(n)} = \mu$, where α is the scattering length for the proton-neutron case and μ is the range which appears in eq. (A.6).

We can now calculate the matrix elements of eq. (6) and (6′). Collecting eq. (6), (A.1), (A.4), (A.6) and taking into account the spherical symmetry of the wave functions φ_q and φ_d (see eqs. (A.2) and (A.5)), we obtain

$$V_{if}^{(p)} = \frac{1}{\sqrt{2}\,v}\left\{2fg\langle\sigma_0\sigma_{1,2}|s_0 s_{1,2}^{(p)}\rangle - FG\langle\sigma_1\sigma_{0,2}|s_0 s_{1,2}^{(p)}\rangle\right\} \cdot \int_{\text{nucleus}} \exp[i(\boldsymbol{k}' - 2\boldsymbol{Q} - \boldsymbol{k}_0 + \boldsymbol{K})\cdot\boldsymbol{r}]\,\mathrm{d}\boldsymbol{r}\,, \tag{A.7}$$

$$V_{if}^{(n)} = \frac{1}{\sqrt{2}\,v}\left\{FG\langle\sigma_2\sigma_{0\,1}|s_0 s_{1,2}^{(n)}\rangle\right\}\int_{\text{nucleus}} \exp[i(\boldsymbol{k}' - 2\boldsymbol{Q} - \boldsymbol{k}_0 + \boldsymbol{K})\cdot\boldsymbol{r}]\,\mathrm{d}\boldsymbol{r}\,, \tag{A.7′}$$

where f, g, F, G are defined as follows:

$$f(\boldsymbol{k}_0 - \boldsymbol{k}') = \int_{\text{all space}} \exp[i(\boldsymbol{k}_0 - \boldsymbol{k}')(\boldsymbol{r}_0 - \boldsymbol{r}_i)]\,V_{0,i}(|\boldsymbol{r}_0 - \boldsymbol{r}_i|)\,\mathrm{d}(\boldsymbol{r}_0 - \boldsymbol{r}_i)\,, \qquad (i = 1 \text{ or } 2),$$

$$g(\boldsymbol{k}_0 - \boldsymbol{k}', q) = \sqrt{v}\int_{\text{all space}} \exp[i(\boldsymbol{k}_0 - \boldsymbol{k}')\cdot\boldsymbol{\rho}/2]\,\varphi_d(\varrho)\,\varphi_q(\varrho)\,\mathrm{d}\boldsymbol{\rho}\,,$$

$$F(\boldsymbol{k}_0 - \boldsymbol{K}/2) = \int_{\text{all space}} \exp[i(\boldsymbol{k}_0 - \boldsymbol{K}/2)\cdot(\boldsymbol{r}_0 - \boldsymbol{r}_i)]\,\varphi_d(|\boldsymbol{r}_0 - \boldsymbol{r}_i|)\,V_{0,i}(|\boldsymbol{r}_0 - \boldsymbol{r}_i|)\cdot \mathrm{d}(\boldsymbol{r}_0 - \boldsymbol{r}_i)\,, \qquad (i = 1 \text{ or } 2),$$

$$G(\boldsymbol{k}' + \boldsymbol{k}_0 - \boldsymbol{K}, q) = \sqrt{v}\int_{\text{all space}} \exp[i(\boldsymbol{k}' + \boldsymbol{k}_0 - \boldsymbol{K})\,\boldsymbol{\rho}/2]\,\varphi_d(\varrho)\,\mathrm{d}\boldsymbol{\rho}\,.$$

Using eq. (A.2′), (A.5) and (A.6) the explicit form of f, g, F, G can be easily calculated and they turn out to be

$$f = \frac{4\pi U}{\mu}\cdot\frac{1}{\mu^2 + |\boldsymbol{k}_0 - \boldsymbol{k}'|^2}\,;$$

$$g = -\frac{8\pi A_d}{\sqrt{\alpha^2 + q^2}\cdot|\boldsymbol{k}_0 - \boldsymbol{k}'|^2}\left\{-\frac{\alpha}{4q}\log\frac{(\alpha_1^2 + \nu_1^2)(\alpha_2^2 + \nu_2^2)}{(\alpha_1^2 + \nu_1^2)(\alpha_2^2 + \nu_1^2)} + \right.$$

$$\left. + \frac{1}{2}\operatorname{arctg}\frac{(\nu_1 + \nu_2)(\alpha_2 - \alpha_1)(\alpha_1\alpha_2 + \nu_1\nu_2)}{(\alpha_1^2 - \nu_1\nu_2)(\alpha_2^2 - \nu_1\nu_2) + \alpha_1\alpha_2(\nu_1 + \nu_2)^2} + \frac{2(\alpha_1 - \alpha_2)(\nu_1 - \nu_2)}{4(\alpha_1 + \mu)(\alpha_2 + \mu) + (\nu_1 + \nu_2)^2}\right\},$$

by putting $\nu_1 = (|\boldsymbol{k}_0 - \boldsymbol{k}'|/2) + q$; $\nu_2 = (|\boldsymbol{k}_0 - \boldsymbol{k}'|/2) - q$;

$$F = \frac{4\pi A_d U}{\mu l}\left(\operatorname{arctg}\frac{l}{\alpha_1 + \mu} - \operatorname{arctg}\frac{l}{\alpha_2 + \mu}\right),$$

by putting $\boldsymbol{l} = \boldsymbol{k}_0 - \boldsymbol{K}/2$;

$$G = \frac{4\pi}{\sqrt{\alpha^2 + q^2}}\left(\frac{1}{\nu^2 - q^2} - \frac{1}{\nu^2 + \mu^2}\right),$$

by putting $\boldsymbol{\nu} = \frac{1}{2}(\boldsymbol{k}' + \boldsymbol{k}_0 - \boldsymbol{K})$.

Appendix B

Average over non observed quantities.

With our units and normalizations, the final state density becomes

(B.1.) $$\varrho_f(K, k') = (2\pi)^{-6}\,\delta(k_0^2 + 2Q^2 + 2q^2 - \tfrac{1}{2}K^2 - k'^2 - 2mB)\,\mathrm{d}\boldsymbol{K}\,\mathrm{d}\boldsymbol{k}',$$

where the δ function represents the energy conservation; B is the binding energy of the nucleons 1 and 2 diminished by the binding energy of the deuteron; m is the nucleon mass. The sum over the non observed quantities, specified by $\boldsymbol{S}$ in eq. (14), consists essentially of the following:

a) the summation (4)(average) over the final (initial) spin states, due to the fact that polarization measurements in the final state are not performed and the incident nucleons are supposed to be unpolarized;

b) the summation over all possible motion states of the nucleon pair in the initial nucleus, *i.e.* in the integration over all possible values of $\boldsymbol{Q}$ and $\boldsymbol{q}$;

c) the integration over all permitted values of $\boldsymbol{k}'$, due to the fact that only the produced deuteron is observed;

d) the sum over the possible pairs of interacting nucleons in the initial nucleus.

Prescription *a*) can be easily performed, remembering the form of the spin wave functions, and the following result is obtained (see eq. (A.7), (A.7′))

(B.2) $$\sum_{\text{spins}} |V_{if}^{(\mathrm{p})}|^2 = \frac{1}{p^2}\,J^{(\mathrm{p})}\cdot\left[\int_{\text{nucleus}} \exp\,[i(\boldsymbol{k}' - 2\boldsymbol{Q} - \boldsymbol{k}_0 + \boldsymbol{K})\cdot\boldsymbol{r}]\,\mathrm{d}\boldsymbol{r}\right],$$

(B.2′) $$\sum_{\text{spins}} |V_{if}^{(\mathrm{n})}|^2 = \frac{1}{p^2}\,J^{(\mathrm{n})}\cdot\left[\int_{\text{nucleus}} \exp\,[i(\boldsymbol{k}' - 2\boldsymbol{Q} - \boldsymbol{k}_0 + \boldsymbol{K})\cdot\boldsymbol{r}]\,\mathrm{d}\boldsymbol{r}\right],$$

by putting

(B.3) $$J^{(\mathrm{p})} = \tfrac{3}{8}(4f^2 g^2 + F^2 G^2 - 2fgFG)\,,$$

(B.3′) $$J^{(\mathrm{n})} = \tfrac{3}{2}F^2 G^2\,.$$

To make the sum indicated in *b*) the distributions of $\boldsymbol{Q}$ and $\boldsymbol{q}$ are needed. We shall assume a gaussian distribution of the type

$$P(\boldsymbol{Q}, \boldsymbol{q})\,\mathrm{d}\boldsymbol{Q}\,\mathrm{d}\boldsymbol{q} = (\pi m\varepsilon)^{-3} \exp\left[-\frac{Q^2+q^2}{m\varepsilon}\right]\mathrm{d}\boldsymbol{Q}\,\mathrm{d}\boldsymbol{q}\,, \tag{B.4}$$

where $\varepsilon = 20$ MeV ([7]).

Eq. (B.4) is equivalent to the assumption of gaussian distributions of $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ (*), *i.e.* by neglecting the correlations between the values of $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$.

The transition probability, summed over the spins, over $\boldsymbol{Q}$, $\boldsymbol{q}$ and $\boldsymbol{k}'$, becomes (see eq. (4), (B.1), (B.2), (B.2′), (B.4))

$$\begin{aligned} w^{(\lambda)}(\boldsymbol{K})\,\mathrm{d}\boldsymbol{K} = {} & (2\pi)^{-5}(\pi m\varepsilon)^{-3} v^{-2}\cdot \\ & \cdot\left\{\int J^{(\lambda)} \exp\left[-\frac{Q^2+q^2}{m\varepsilon}\right]\left[\int\limits_{\text{nucleus}} \exp\left[i(\boldsymbol{k}'-2\boldsymbol{Q}-\boldsymbol{k}_0+\boldsymbol{K})\cdot\boldsymbol{r}\right]\mathrm{d}\boldsymbol{r}\right]^2\cdot\right. \\ & \left.\cdot\,\delta(k_0^2+2Q^2+2q^2-\tfrac{1}{2}K^2-k'^2-2mB)\,\mathrm{d}\boldsymbol{Q}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{k}'\right\}\mathrm{d}\boldsymbol{K}\,. \end{aligned} \tag{B.5′}$$

To simplify the integration we shall use the following approximation ([11]):

$$\left[\int\limits_{\text{nucleus}} \exp\left[i(\boldsymbol{k}'-2\boldsymbol{Q}-\boldsymbol{k}_0+\boldsymbol{K})\cdot\boldsymbol{r}\right]\mathrm{d}\boldsymbol{r}\right]^2 \simeq (2\pi)^3 v\,\delta(\boldsymbol{k}'-2\boldsymbol{Q}-\boldsymbol{k}_0+\boldsymbol{K})\,, \tag{B.6}$$

which can be obtained by performing the integration in a square box of side $2a = v^{\frac{1}{3}}$ and remarking that

$$\delta(k) = \lim_{a\to\infty} \frac{1}{a\pi}\,\frac{\sin^2 ak}{k^2}\,.$$

The introduction of eq. (B.6) into eq. (B.5′) eliminates the integration over $\boldsymbol{Q}$ and makes easy the integration over the angular co-ordinates of $\boldsymbol{k}'$ and $\boldsymbol{q}$. We get

$$\begin{aligned} w^{(\lambda)}(\boldsymbol{K})\,\mathrm{d}\boldsymbol{K} = {} & 2(\pi m\varepsilon)^{-3} v^{-1}\cdot \\ & \cdot\left\{\int J^{(\lambda)}(\nu^2 = l^2+q^2-mB)\exp\left[-x\right] I_0(y)\,P^2 q^2\,\mathrm{d}P\,\mathrm{d}q\right\}\mathrm{d}\boldsymbol{K}\,. \end{aligned} \tag{B.7}$$

$J^{(\lambda)}(\nu^2 = l^2+q^2-mB)$ means that the $J^{(\lambda)}$ have to be calculated for $\nu^2 = l^2+q^2-mB$ (see the expression of the G function, given in Appendix A); $I_0(y)$ is the Bessel function of zero order defined by

$$I_0(y) = \frac{1}{2\pi}\int_0^{2\pi} \exp\,(y\cos\varphi)\,\mathrm{d}\varphi = \sum_{m=0}^{\infty} (m\,!)^{-2}(y/2)^{2m};$$

(*) $P(\boldsymbol{k}_i) = (2\pi m\varepsilon)^{-\frac{3}{2}} \exp\left[-k_i^2/2m\varepsilon\right]$.

([11]) B. A. Lippmann and J. Schwinger: *Phys. Rev.*, **79**, 469 (1950).

x and y are respectively

$$x = (4m\varepsilon)^{-1}\{4q^2 + P^2 + K^2 - l^{-1}K\cos\theta\,(\tfrac{1}{2}P^2 - 2q^2 + 2mB)\}\,,$$
$$y = (4m\varepsilon)^{-1}l^{-1}K\sin\theta\,\{4l^2P^2 - (\tfrac{1}{2}P^2 - 2q^2 + 2mB)^2\}^{\frac{1}{2}}\,,$$

where $P = |\boldsymbol{k}_0 - \boldsymbol{k}'|$ and $\theta = \widehat{\boldsymbol{k}_0\boldsymbol{K}}$; the integration limits for q and P are fixed by the energy conservation and are

$$\sqrt{mB - l^2} \leqslant q \leqslant \infty\,, \qquad \text{if } l^2 < mB,$$
$$0 \leqslant q \leqslant \infty\,, \qquad \text{if } l^2 > mB,$$

$$\text{Max}\,\{2(l-\xi);\ 2(-l+\xi)\} \leqslant P \leqslant 2(l+\xi)\,, \qquad \text{being} \qquad \xi = \sqrt{l^2 - mB + x^2}\,.$$

The integrations over q and P have to be done numerically.

In order to obtain the final expression of the transition probability the summation over all possible interacting nucleon pairs is still to be performed. The way to do this is not obvious, as it requires some further hypothesis about the quasi-deuteron nuclear model. Further the result is sensitive to the incident nucleon and final deuteron absorption effects, which have not been taken into account. The relative importance of the processes in which particle 1 is a proton ($\lambda = p$) or a neutron ($\lambda = n$) can be estimated by observing that the ratio between the numbers of neutron-neutron and neutron-proton pairs turns out to be (assuming charge independence), $N/8Z$ (*) (N, Z = neutron, proton number). So the transition probability can be written

$$w(\boldsymbol{K}) = C(A)\left[w^{(\mathrm{p})}(\boldsymbol{K}) + \frac{N}{8Z}\,w^{(\mathrm{n})}(\boldsymbol{K})\right], \tag{B.7'}$$

where the constant C, depending on the mass number A, should be determined empirically.

Using eq. (B.3) and (B.5), eq. (B.7') becomes

$$w(\boldsymbol{K})\,\mathrm{d}\boldsymbol{K} = C(A)\cdot(\pi m\varepsilon)^{-3}\ \Gamma\,\mathrm{d}\boldsymbol{K}\,, \tag{B.7''}$$

where

$$\Gamma = \int J(\nu^2 = l^2 + q^2 - mB)\exp[-x]\,I_0(y)\,P^2q^2\,\mathrm{d}P\,\mathrm{d}q\,, \tag{B.8}$$

and

$$J \equiv \frac{3}{4}\left[4f^2g^2 + \left(1 + \frac{N}{2Z}\right)F^2G^2 - 2fgFG\right].$$

(*) $\varphi_q(\varrho)$ is assumed to be an s wave. More generally, the same result is obtained assuming a Serber interaction.

The differential cross section, as a function of the energy E_d of the produced deuteron and of the angle θ between $\boldsymbol{K}$ and $\boldsymbol{k}_0$ ($\boldsymbol{k}_0$ is a symmetry axis) can finally be written

$$\sigma(E_d, \theta)\,\mathrm{d}E_d\,\mathrm{d}\Omega = \frac{C(A)}{v}\cdot\frac{2}{(\pi\varepsilon)^3 m}\cdot\frac{K}{k_0}\,\Gamma\,\mathrm{d}E_d\,\mathrm{d}\Omega\,. \tag{B.9}$$

In the numerical computations of the cross section the following choice of the parameters was made ([1,5,8]):

$A_d^2 = 0.0613\cdot 10^{13}$ cm^{-1}; $\alpha_1 = 0.231\cdot 10^{13}$ cm^{-1}; $\alpha_2 = 1.55\cdot 10^{13}$ cm^{-1};

$U = 67.8$ MeV; $\mu = 0.847\cdot 10^{13}$ cm^{-1}; $\alpha^{-1} = 5.39\cdot 10^{-13}$ cm;

$B = 40$ MeV; $\varepsilon = 20$ MeV; $\dfrac{\hbar k_0^2}{2m} = 120$ MeV.

RIASSUNTO

Cerchiamo di interpretare la produzione di deutoni il cui spettro d'energia e la distribuzione angolare non possono spiegarsi con un processo di « pick-up » sulla base di un modello nucleare a particelle indipendenti. Abbiamo usato il modello nucleare a due-nucleoni, o quasi-deutonico, trovando un ragionevole accordo con i dati sperimentali.

IL NUOVO CIMENTO Vol. XVIII, N. 4 16 Novembre 1960

Limites sur les interactions anormales des muons et leurs manifestations dans la diffusion nucléaire à haute énergie.

B. Jouvet et L. Goldzahl

Laboratoire de Physique Atomique et Moléculaire, Collège de France - Paris

(ricevuto il 25 Luglio 1960)

Résumé. — On détermine à partir de données expérimentales récentes, (atomes mésiques, moment magnétique), des limites pour les masses et constantes de couplage de bosons, scalaire, vectoriel et pseudoscalaire, susceptibles d'expliquer la masse anormale du muon. On discute ensuite des conditions d'une mise en évidence d'une interaction anormale muon-nucléon dans la diffusion élastique sur les noyaux.

Introduction.

Quoique le plus ancien des mésons à avoir été découvert, le muon est devenu la particule la plus mystérieuse.

En effet, les mesures effectuées durant ces deux dernières années montrent que, à part la masse qui le différencie de l'électron, le lepton μ possède, aux incertitudes expérimentales près, des interactions avec les autres champs identiques à celles de l'électron,

— même charge,

— même moment magnétique anormal correspondant à un couplage purement électromagnétique,

— couplage symétrique de μ et e avec les neutrinos,

— couplage symétrique μ-ν et e-ν avec les nucléons,

— même couplage (fondamental pseudovectoriel, ou secondaire) avec les pions.

Selon la conception des particules élémentaires dans le schéma de la théorie des champs, cette différence de masse pourrait être expliquée par l'une des hypothèses suivantes:

1) ou bien le muon possède par rapport à l'électron la propriété d'interagir avec quelque champ qui n'interagit que beaucoup plus faiblement ou pas du tout avec les électrons,

2) ou bien les masses nues des électrons et des muons sont fondamentalement différentes, ce qui n'a de signification expérimentale qu'à des énergies énormes,

3) ou bien si on admet que le rayon de coupure qu'on doit introduire dans la théorie des perturbations pour le calcul de la self-masse a quelques sens physique, il se peut aussi que les rayons «fondamentaux» de ces deux particules soient différents. Cependants si ce type de rayon a un sens mathématique relativement bien défini, il n'a pour l'instant aucun sens physique bien défini, ni conceptuellement ni expérimentalement, car on sait par l'exemple de la différence de masse des nucléons, que ce rayon ne correspond pas au rayon électromagnétique observé dans les expériences de Hofstadter.

Du point de vue expérimental, les nouvelles propriétés du muon qui pourraient éventuellement être mises en évidence dans des expériences de diffusion des muons sur des noyaux sont:

a) un rayon électromagnétique du lepton μ différent de celui de l'électron

b) une interaction anormale avec les nucléons.

L'une ou l'autre de ces manifestations pourrait être interprétée par une explication de type (1).

On se propose dans le présent article:

1) de fixer, à la lumière des expériences faites jusqu'à ce jour et compte tenu de leur marge d'erreur, des limites maximales

a) pour les interactions anormales muon-nucléon,

b) pour le rayon électromagnétique du muon,

c) pour les interactions anormales muon-muon,

2) de voir dans quelle mesure des expériences de diffusion élastique du muon sur des noyaux peuvent permettre d'apporter des informations plus précises que cells que nous avons déjà.

Certes, des expériences de diffusion inélastiques peuvent aussi apporter d'autres informations. Cet aspect ne sera pas discuté pans le présent article.

Chapitre I.

1. – Limitation sur l'existence d'interaction anormale muon-nucléon.

Nous appelons interaction anormale, l'interaction qui se différencie de celles déjà connues et que nous supposons théoriquement identiques pour l'électron et le muon.

Nos informations sur l'interaction muon-nucléon résultent essentiellement:

1) *Des diffusions muon-noyau faites avec les muons des rayons cosmiques.*

Certaines expériences faites avec les rayons cosmiques ont semblé montrer que la diffusion des mésons présente certaines anomalies [1]. Cette diffusion anormale aurait les caractéristiques suivantes:

— elle ne commencerait à paraître que pour des impulsions des mésons supérieures à 600 MeV/c; par ailleurs, elle varierait en fonction de la charge nucléaire comme Z^2, ce qui conduit à penser qu'elle se manifeste principalement dans la diffusion élastique et qu'elle provient d'une interaction anormale des muons avec les protons.

D'autres expériences n'ont pas confirmé ces anomalies [2] et l'on a pu penser que les anomalies de diffusion constatées seraient dûes essentiellement à une mésestimation des corrections de diffusion multiple. En tout état de cause, les calibrages des énergies des muons et les statistiques ne permettent pas de lever l'incertitude sur la valeur de ces expériences; aussi la seule conclusion que l'on puisse valablement tirer est que si un effet anormal existe, sa section efficace doit être au plus de l'ordre de 10^{-28} cm^2 par nucléon pour des énergies > 200 MeV et des angles $> 10°$.

2) *De l'étude des atomes mésiques.*

Toute interaction non électromagnétique des mésons μ avec le noyau produit un déplacement anormal des niveaux des atomes mésiques. Comme l'a remarqué Wheeler [3], la grande valeur des fonctions d'ondes du muon dans le domaine nucléaire, comparées à celles de l'électron, rend en principe très favorable pour l'étude des interactions noyaux-muon, la mesure des niveaux atomiques. Mais l'imprécision sur la mesure des énergies des photons émis

(1) A. W. Wolfendale: *Rev. Mod. Phys.*, **30**, 441 (1958); P. H. Fowler et A. W. Wolfendale: *Progress in Cosmic Ray Phys.*, vol. **4** (Amsterdam, 1958), p. 123; J. L. Lloyd et A. W. Wolfendale: *Phys. Rev.*, **117**, 247 (1960).

(2) S. Fukui, T. Kitamura and Y. Watase: *Phys. Rev.*, **113**, 315 (1959). Cet article contient aussi de nombreuses références bibliographiques.

(3) J. A. Wheeler: *Rev. Mod. Phys.*, **21**, 133 (1949).

par les atomes mésiques ainsi que l'incertitude sur la valeur de la masse du muon limitent pour l'instant les informations qu'on peut tirer de cette méthode.

Cependant, les mesures de Koslov [4] sur toute une série d'atomes ont montré que:

1) il n'y a pas d'interaction anormale importante,

2) l'interaction muon-noyau résultant de la polarisation du vide par les paires d'électrons est nettement mise en évidence.

Les résultats de Koslov que nous allons utiliser sont résumés dans le tableau suivant.

Tableau I.

transition	valeurs calculées de m_μ	
	sans polarisation du vide	avec polarisation du vide
C: $2p$-$1S$	< 209.99	< 208.95
P: $3D$-$2P$	> 207.67	> 206.89
Sc: $4F$-$2D$	> 206.82	> 206.47

Par ailleurs, de récentes mesures directes (*) de m_μ par Lathrop *et al.*, et Bearden *et al.* [5] (**) fournissent:

$$206.74 < \frac{m_\mu}{m_e} < 206.79 \qquad \text{donc} \qquad \delta m_\mu = 0.05\ \mathrm{m_e}\,.$$

Nous allons voir quelles limites ces données apportent sur l'existence d'une interaction anormale.

[4] S. Koslov: *Nevis Report* no. 19 (Columbia, 1956), non publié; S. Koslov, V. L. Fitch et J. Rainwater: *Phys. Rev.*, **95**, 291, 625 (1954).

(*) *Note ajoutée à la correction des épreuves.* - Contrairement à ce que laisse penser la brève information rapportée par Garwin *et al.* (réf. [7]), dans leur *Note added in proof*, les mesures de Lathrop *et al.* et Bearden (réf. (**)) sont une amélioration des mesures de Koslov sur le phosphore, et non une mesure directe de m_μ. Ainsi le désaccord, dont nous parlons dans cet article, entre la limite inférieure de m_μ obtenue par Koslov sur P, et la limite supérieure obtenue par Lathrop *et al.*, n'existe pas.

[5] Valeurs rapportées par Garwin, D. P. Hutchinson, S. Penman and G. Shapiro: réf. [7].

(**) J. Lathrop, R. A. Lundy, V. L. Telegdi, R. Winston and D. D. Yovanovitch: *Nuovo Cimento*, **17**, 109 (1960); J. Lathrop, R. A. Lundy, S. Penman, V. L. Telegdi, R. Winston et D. D. Yovanovitch; A. J. Bearden: *Nuovo Cimento*, **17**, 114 (1960).

Si nous supposons qu'il existe une interaction anormale muon-nucléon celle-ci peut être généralement déduite de la donnée d'un terme de couplage en principe non local entre les nucléons et les muons.

Si l'interaction résulte de l'existence de couplages de champs de masse M_i de spin zéro (scalaire ou pseudoscalaire), de spin 1 (vectoriel ou pseudovectoriel), dont nous supposons que les couplages hermitiens

$$(G_i^{\mathcal{N}}\overline{\psi}_{\mathcal{N}}\Gamma_i\psi_{\mathcal{N}} + G_i^{\mu}\overline{\psi}_{\mu}\Gamma_i\psi_{\mu})\varphi_i\,, \qquad \Gamma_i = 1,\ i\gamma_{\mu},\ i\gamma_{\mu}\gamma_5,\ i\gamma_5\,,$$

ne font pas intervenir de dérivées, elle est à basse énergie, analogue à l'une des interactions du type de Fermi, faisant intervenir les covariants usuels: S, V, A et P, avec des constantes d'interactions g_S, g_V, g_P, telles que $g_i = G_i^{\mathcal{N}} G_i^{\mu}/M_i$, pourvu que $M_i \ll \alpha m_{\mu} \sim m_e$.

Il reste cependant, pour obtenir l'interaction muon-noyau, à postuler quelles relations existent entre les interactions muon-neutron et muon-proton.

Selon que l'interaction n'a lieu qu'avec les protons, avec les neutrons symétriquement ou antisymétriquement, la constante de couplage muon-noyau sera

$$hg = (gZ,\ gN,\ gA.\ g(Z-N))\,.$$

Si les masses M_i sont $\geqslant 2m_{\pi}$, le domaine effectif d'interaction des muons avec les noyaux reste sensiblement le même que dans le cas de l'interaction de polarisation du vide et le déplacement anormal des niveaux fait intervenir le même type d'intégrale sur les fonctions d'onde du muon.

On a alors pour ce déplacement

$$(\Delta E)_{\text{anormal}} \approx (g_S + g_V)h\int_V \psi^*\psi\,\mathrm{d}v + (g_A + \Gamma_S)h(S\cdot s)\int_V \psi^*\psi\,\mathrm{d}v\,,$$

où $(\boldsymbol{S}\cdot\boldsymbol{s})$ est la valeur moyenne du produit des spins du noyau et du muon et $\Gamma_S \approx (\alpha^2/4)(m_{\mu}/AM_{\mathcal{N}})g_P$, la constante effective d'interaction locale résultant du couplage pseudoscalaire.

Par ailleurs, le déplacement de niveau produit par la polarisation du vide au 1$^{\text{er}}$ ordre en α est donné par l'expression analogue à la précédente,

$$(\Delta E)_{\text{polarisation}} \approx \frac{Ze^2}{m_e^2}\,\frac{\alpha}{15\pi}\int_V \psi^*\psi\,\mathrm{d}v\,, \qquad \alpha = \frac{e^2}{4\pi} = \frac{1}{137}\,.$$

Nous voyons d'une part, d'après le Tableau I, que dans le cas le plus favorable de C $(2P-1S)$ la polarisation du vide produit une variation de masse mesurée de $\Delta m_{\mu} \approx 1\ \mathrm{m}_e$.

Puisque, quelle que soit la précision mesurée des niveaux atomiques la masse n'est connue qu'à $5/100\ m_e$ près, il est clair qu'une interaction anormale équivalente à $5/100$ de celle de polarisation du vide ne peut être mise en évidence actuellement dans les atomes mésiques. (Remarquons que l'interaction de polarisation du vide de second ordre n'est que $41/162(15/\pi)\alpha \approx 1/100$ fois celle du premier ordre.)

Donc la constante g est limitée par (*)

$$|gh| < \frac{5}{100}\,\frac{4\pi Z\alpha}{m_e^2}\,\frac{\alpha}{15\pi} \simeq 7Z10^{-7}\,\frac{1}{m_e^2},$$

soit (**)

$$|g| < 7\cdot 10^{-7}\,\frac{1}{m_e^2} \simeq 1.15\cdot 10^{-27}\ \mathrm{cm}^2.$$

D'autre part, s'il faut accepter la valeur donnée par KOSLOV pour les niveaux de P, il devrait exister une interaction anormale qui fasse décroître la valeur calculée d'au moins $(206,\ 89 - 206,\ 79) = 0.1\ m_e$ (***).

Puisque pour les niveaux considérés la polarisation du vide produit un changement de $0.78\ m_e$, il faudrait une interaction anormale de même signe que la polarisation du vide, c'est-à-dire attractive, et d'intensité 1/7.8 fois plus petite, soit $gh = Z\,18\cdot 10^{-7}\,1/m_e^2$.

2. – Limites sur le rayon électromagnétiques du muon.

Un effet observable de la structure électromagnétique du muon est la modification de la valeur de son moment magnétique anormal.

Nous allons utiliser un calcul de BERESTETSKII (6) qui a estimé l'effet sur le moment magnétique d'une modification de la forme des équations de l'électrodynamique à petite distance, et obtenu pour le moment magnétique anormal $\delta\mu$

$$\frac{\delta\mu}{\mu_0} = \frac{\alpha}{2\pi}\left(1 - \frac{2}{3}\left(\frac{m_\mu}{\Lambda}\right)^2\right).$$

(*) On a pris $\hbar = c = 1$; alors $1/m_e \simeq 4\cdot 10^{-11}$ cm.

(**) Comparant les mesures de V. L. FITCH et J. REINWATER (*Phys. Rev.*, **52**, 789 (1953)), aux calculs de L. N. COOPER et E. M. HENLEY (*Phys. Rev.*, **92**, 801 (1953)) dans lesquels les corrections de polarisation du vide sont omises, I. R. GATLAND (*Nucl. Phys.*, **14**, 205 (1959)) obtient, en prenant $(h = \Lambda)$, les limites sur la constante g, telle qu'elle est definie ici $g \leqslant 3.6\cdot 10^{-7}/m_e^2$ et $3.6\cdot 10^{-8}/m_e^2$ pour $Z = 13$ et 29.

(***) Voir note (*) à la p. 705.

(6) V. B. BERETETSKII, O. N. KROKHIN et A. K. KHLEBNIKOV: *Žurn. Eksp. Teor. Fiz.*, **30**, 788 (1956).

m_μ est la masse du muon et Λ une constante de la dimension d'une masse introduite dans le propagateur de photon dont la forme est supposée être

$$\frac{1}{p^2}\,\frac{\Lambda^2}{p^2+\Lambda^2}\,.$$

Soit en effet $F(q^2)$ le facteur de structure électromagnétique du méson; nous le supposerons de la forme $F_n(q^2) \cong (\Lambda^2/(\Lambda^2+q^2))^n$ où n est une puissance entière ou demi-entière $\geqslant \frac{1}{2}$. Pour q^2 petit, on a $F_n(q^2) \simeq 1 - n(q^2/\Lambda^2)$.

Le rayon électromagnétique quadratique moyen du méson (r_0^2) s'obtient par le développement de $F(q^2)$ pour q^2 petit par la relation

$$F(q^2) \simeq 1 - \tfrac{1}{6}\langle r_0^2\rangle q^2\,,$$

donc

$$\langle r_0^2\rangle = \frac{1}{\Lambda^2}\,6n\,.$$

Avec le facteur de structure F_n postulé plus haut, le propagateur effectif qui intervient dans l'intégrale dont le calcul donne le moment magnétique, est

$$\frac{1}{q^2}\,F_n^2(q^2) \quad \frac{1}{q^2}\left(\frac{\Lambda^2}{q^2+\Lambda^2}\right)^{2n} \quad (-1)^{2n-1}\,\frac{(\Lambda^2)^{2n}}{(2n-1)!}\,\frac{\partial^{2n-1}}{(\partial\Lambda^2)^{2n-1}}\left(\frac{1}{q^2}\,F_{\frac{1}{2}}^2(q^2)\,\frac{1}{\Lambda^2}\right),$$

au lieu du propagateur $(1/q^2)F_{\frac{1}{2}}^2(q^2)$ utilisé per Beretetskii.

Pour un n donné on a

$$\frac{\delta\mu}{\mu_0} = (-1)^{2n-1}\,\frac{(\Lambda^2)^{2n}}{(2n-1)!}\,\frac{\partial^{2n-1}}{(\partial\Lambda^2)^{2n-1}}\left(\frac{\alpha}{2\pi}\left(\frac{1}{\Lambda^2}-\frac{2}{3}\,\frac{m_\mu^2}{(\Lambda^2)^2}\right)\right) =$$

$$\frac{\alpha}{2\pi}\left(1-\frac{2}{3}\cdot 2n\,\frac{m_\mu^2}{\Lambda^2}\right) - \frac{\alpha}{2\pi}\left(1-\frac{2}{9}\,m_\mu^2\,\langle r_0^2\rangle\right),$$

résultat indépendant de la puissance n choisie (*).

Les mesures de Garwin *et al.* (7) donnent pour limite inférieure du facteur de Landé du muon

$$g \geqslant 2(1.00122 \pm 0.00008) \geqslant 2(1.00114)\,.$$

On a d'autre part la limite supérieure (5)

$$g \leqslant 2\,(1.00129)\,.$$

(*) Résultat communiqué par Mr. M. Mathieu.

(7) R. L. Garwin, D. P. Hutchinson, S. Penman and G. Shapiro: *Phys. Rev. Lett.*, **2**, 516 (1959); *Phys. Rev.*, **118**, 271 (1960).

Sans rayon électromagnétique ni interaction anormale, la valeur théorique est (8)

$$g_{\text{th}} = 2\left(1 + \frac{\alpha}{2\pi} + 0.75\,\frac{\alpha^2}{\pi^2}\right) = 2(1.0011654)\,.$$

Si le rayon électromagnétique doit par lui seul rendre compte de cette différence entre la valeur théorique et la limite expérimentale, il doit être tel que

$$\frac{\alpha}{2\pi}\,\frac{2}{9}\,m_\mu^2\langle r_0^2\rangle \leqslant 1.001165 - 1.00114 = 25\cdot 10^{-6}\,.$$

Soit

$$r_0 m_\mu \leqslant 0.3\,, \qquad r_0 \leqslant 0.6\cdot 10^{-13}\ \text{cm}\,. \qquad (r_0 = \sqrt{\langle r_0^2\rangle})\,.$$

Cette limite est environ deux fois plus grande que la limite extimée pour le rayon électromagnétique de l'électron.

3. – Limites sur les interactions anormales muon-muon.

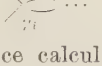
Cependant, on ne peut guère s'attendre à ce que le muon puisse avoir un rayon anormal par rapport à celui de l'électron, sans qu'il ne possède une interaction anormale qui induise ce rayon. Mais une telle interaction donne aussi une contribution directe au moment magnétique.

Si G_S^μ, G_V^μ, G_P^μ, M_S, M_V, M_P caractérisent les constantes de couplage et les masses de champs intermédiaires supposés neutres scalaire, vectoriel ou pseudoscalaire (*), les moments magnétiques anormaux résultant sont (**) (9),

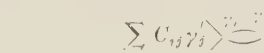
(8) H. Suura et K. Wichmann: *Phys. Rev.*, **105**, 1930 (1957); A. Petermann: *Phys. Rev.*, **105**, 1931 (1957).

(*) Nous ne considérerons ici que les trois cas qui ne soulèvent pas de difficulté de renormalisation et ne nécessitent pas l'introduction de cut-off arbitraires.

(**) Il est intéressant de noter les différences de signe ainsi que les dépendances différentes en M/m_μ pour de grandes valeurs de ce paramètre, de ces expressions. Ce fait se met en évidence très simplement de la façon suivante: si $M_i \to \infty$ l'interaction produite par le boson (i) tend vers une interaction de contact d'intensité (G_i^2/M_i^2) et le graphe calculé tend vers le graphe



Faisant une transformation de Fierz, ce calcul est équivalent au calcul des graphes



dont le seul terme contenant le tenseur $\gamma_j' = \gamma_T = \gamma_\mu\gamma_\nu$ est à considérer; on voit alors que seules les interactions non locales de type scalaire, tenseur et pseudoscalaire, produisent un moment magnétique qui, à la limite $M^2 \to \infty$, est au plus de l'ordre de $(m_\mu^2/M^2)\log(M^2/m_\mu^2)$. Les différences de signes entre S et P proviennent du facteur i du couplage pseudoscalaire hermitien.

(9) W. S. Cowland: *Nucl. Phys.*, **8**, 397 (1958); S. N. Gupta: *Phys. Rev.*, **111**, 1436 (1958); B. De Tollis: *Nuovo Cimento*, **14**, 253 (1959).

en supposant que $M_i \ll m_\mu$,

$$\delta\mu_P/\mu_0 = -\left((G_P^\mu)^2/4\pi\right)\cdot\frac{1}{2\pi}\left(m_\mu^2/M_P^2\right)\log\left(M_P^2/m_\mu^2\right),$$

$$\frac{\delta\mu_S}{\mu_0} = +\frac{(G_S^\mu)^2}{4\pi}\frac{1}{2\pi}\frac{m_\mu^2}{M_S^2}\log\left(\frac{M_S^2}{m_\mu^2}\right); \qquad \frac{\delta\mu_V}{\mu_0} = +\frac{(G_V^\mu)^3}{4\pi}\frac{1}{\pi}\left(\frac{1}{3}\frac{m_\mu^2}{M_V^2} - \frac{m_\mu^4}{M_V^4}\log\left(\frac{M_V^2}{m_\mu^2}\right)\right).$$

On a donc en général

$$\frac{\delta\mu}{\mu_0} = \frac{\alpha}{2\pi}\left(1-\frac{2}{9}m_\mu^2\langle r_0^2\rangle\right) + \frac{1}{\pi}\frac{(G_V^\mu)^2}{4\pi}\left(\frac{1}{3}\frac{m_\mu^2}{M_V^2}\right) + \left(\frac{G_S^{\mu 2}}{4\pi}\right)\frac{1}{2\pi}\frac{m_\mu^2}{M_S^2}\log\left(\frac{M_S^2}{m_\mu^2}\right) - \frac{(G_P^\mu)^2}{4\pi}\frac{1}{2\pi}\frac{m_\mu^2}{M_\mu^2}\log\left(\frac{M_P^2}{m_\mu^2}\right).$$

Donc, selon le type et les valeurs des couplages envisagés, on déduit de la limite expérimentale différentes bornes pour le rayon. Mais, par ailleurs, la valeur du rayon électromagnétique dépend de l'existence d'interaction anormale.

Les rayons électromagnétiques qu'on estime facilement d'après (*) sont de l'ordre de

$$\frac{(G_i^\mu)^2}{4\pi}\frac{1}{2\pi}\frac{1}{M_i^2}\log\frac{\Lambda_i^2}{m_\mu^2}.$$

Il est alors certain que si un effet anormal doit apparaître dans la valeur expérimentale du moment magnétique, il proviendra davantage de l'effet direct de l'interaction anormale sur le moment magnétique que de l'effet indirect du rayon du muon dans le moment magnétique, à moins, bien sûr, que cet effet ne soit dû à une violation des lois de l'électrodynamique.

La borne inférieure du moment magnétique limite alors une interaction muon-muon de type pseudo-scalaire, si elle est la seule à intervenir comme facteur anormal, à satisfaire l'inégalité

$$\frac{(G_P^\mu)^2}{4\pi}\frac{m_\mu^2}{M_P^2}\log\left(\frac{M_P^2}{m_\mu^2}\right) \leqslant 2.5\cdot 10^{-5}\cdot 2\pi = 15\cdot 10^{-4}.$$

Au contraire la limite supérieure du moment magnétique fournit pour l'interaction vectorielle ou scalaire seule agissante entre les muons, les bornes

$$\frac{(G_V^\mu)^2}{4\pi}\left(\frac{m_\mu}{M_V}\right)^2 \leqslant 3\pi\cdot 13\cdot 10^{-4} = 1.2\cdot 10^{-2},$$

$$\frac{(G_S^\mu)^2}{4\pi}\left(\frac{m_\mu}{M_S}\right)^2\log\left(\frac{M_S^2}{m_\mu^2}\right) \leqslant 0.8\cdot 10^{-2}$$

(*) Voir note (**) à page précédente.

Si l'on admet que muon et électron ont même masse nue, et que leur masse est calculable en théorie de perturbation en utilisant un cut-off universel, on devrait avoir (*)

$$\frac{m_\mu}{m_e} \approx \frac{G_\mu^2}{e^2} \qquad \text{d'où} \qquad \frac{G_\mu^2}{4\pi} \approx \frac{200}{137} \approx 1.5 .$$

On aurait alors:

$$M_S \geqslant 4M , \qquad M_V \geqslant 1.2M , \qquad M_P \geqslant 350M \qquad (M = \text{masse du nucléon}).$$

4. – Conclusion.

Des deux limites données à la Sec. 1'2 sur l'interaction muon-nucléon, nous pouvons alors déduire des inégalités pour les constantes de couplage entre nucléon et boson scalaire, pseudoscalaire ou vectoriel:

— Si nous admettons que le déplacement anormal des niveaux du phosphore est réel, on a alors pour les constantes $G_S^{\mathcal{N}}$, $G_V^{\mathcal{N}}$, $G_P^{\mathcal{N}}$ d'interaction du boson scalaire, pseudoscalaire, ou vectoriel avec les nucléons, les limites (**)

$$\frac{(G_S^{\mathcal{N}})^2}{4\pi} \gtrsim 40; \qquad \frac{(G_V^{\mathcal{N}})^2}{4\pi} \gtrsim 0.32; \qquad \frac{(G_P^{\mathcal{N}})^2}{4\pi} \gtrsim 5 \cdot 10^{20} .$$

— Si, au contraire, on admet que l'on a seulement une limite supérieure de l'interactionm muon-nucléon on a alors des relations du type

$$M_i > \frac{C(i)}{\sqrt{\delta\mu}} , \qquad \frac{(G_i^{\mathcal{N}})^2}{4\pi} < C_0 M_i^4 (\delta g_{\mu\mathcal{N}})^2 .$$

$\delta\mu$ et $\delta g_{\mu,\mathcal{N}}$ étant respectivement les limites d'erreur sur les moment magnétique du muon et son interaction avec les nucléons, $C(i)$ des constantes dépendant du type tensoriel envisagé et faiblement variable avec M_i, et C_0 une constante.

Par exemple, pour $g_{\mu\mathcal{N}} \simeq 7 \cdot 10^{-7}\ 1/\mathrm{m}_e^2$, limite actuelle d'imprécision, on aurait

$$\frac{(G_S^{\mathcal{N}})^2}{4\pi} \gtrsim 8.4 , \qquad \frac{(G_V^{\mathcal{N}})^2}{4\pi} \gtrsim 0.05 .$$

(*) Il est remarquable que ce type de relation donne un résultat acceptable pour la masse du nucléon, alors que la constante de couplage $G^2/4\pi \sim 15$ ne permet plus de la justifier.

(**) Voir note (*) p. 705.

Bien que notre analyse ne soit que fragmentaire en raison des limitations que nous avons apportées au choix immense des couplages possibles, ces chiffres nous montrent l'importance de toute mesure précise du moment magnétique d'une part et de l'interaction muon-nucléon d'autre part. En effet, les premières mesures apportent des renseignements essentiels sur des champs de très grande masse pour lesquelles il est difficile d'obtenir des informations directes étant donné leur très court temps de vie et leur petit domaine d'interaction. Par ailleurs, une mise en évidence certaine de l'interaction muon-nucléon permet alors de déterminer la constante de couplage boson-nucléon. Bien qu'il ne semble pas possible sur la base d'arguments expérimentaux de rejeter sans étude plus poussée l'existence de couplages très intenses entre nucléons lorsque le domaine d'interaction est très inférieur au rayon du cœur des nucléons, il résulte des valeurs précédentes que l'existence d'une interaction vectorielle plutôt que scalaire serait mieux accueillie par les théoriciens!

Au sujet de la valeur importante trouvée pour $(G_S^N)^2$ (sans parler de $(G_P^N)^2$!) il convient d'une part de remarquer qu'il s'agit d'une constante définie au sens de Kroll et Ruderman ($p^2 = 0$), et qui peut différer très notablement de celle définie par Watson ($p^2 = -M_S^2$) et d'autre part que cette valeur n'invalide pas la théorie de perturbation pour les processus que nous avons étudiés, dans, lesquels intervient l'échange d'un seul méson assez faiblement couplé aux muons. Il en serait tout autrement si nous devions étudier l'interaction nucléon-nucléon résultant de ces couplages.

Chapitre II.

1. – Effet du rayon électromagnétique du méson μ dans la diffusion par des noyaux.

Le proton, le muon et l'électron étant doués d'une extension caractérisée respectivement par les facteurs de structure $F_P(q^2)$, $F_\mu(q^2)$, $F_e(q^2)$, la section efficace de diffusion d'un muon ou d'un électron avec un proton peut s'écrire à l'approximation de Born

$$\left(\frac{d\sigma}{d\Omega}\right)_{P/\mu,e} = F_P^2(q^2)\, F_{\mu,e}^2(q^2) \left(\frac{d\sigma}{d\Omega}\right)_{point},$$

q^2 est le module relativiste de la quadri-impulsion échangée dans la diffusion: $(q^2 = (2p \sin\theta/2)^2)$, p est le module de l'impulsion du lepton incident, θ est l'angle de diffusion dans le laboratoire.

Pour des valeurs de q^2 petites ($\leqslant (200)^2 \frac{1}{4}$ MeV $\sim 10^4$ MeV)

$$F_{\mu,e}(q^2) \simeq 1 - \tfrac{1}{6} \langle r_0^2 \rangle_{e,\mu}\, q^2 .$$

Puisque par les expériences de diffusion de type d'Hofstadter le rayon de l'électron n'est pas connu, mais seulement la somme du rayon quadratique du proton et de l'électron, il est indiqué de comparer les sections efficaces de diffusion de l'électron ou du muon.

On a:

$$\frac{(\mathrm{d}\sigma/\mathrm{d}\Omega)_{P/\mu} - (\mathrm{d}\sigma/\mathrm{d}\Omega)_{P/e}}{(\mathrm{d}\sigma/\mathrm{d}\Omega)_{P/e}} \approx \tfrac{1}{3}(\langle r_0^2\rangle_\mu - \langle r_0^2\rangle_e) q^2 .$$

Pour mesurer cette expression, il faut qu'elle soit plus grande que la somme des erreurs faites sur les mesures de section efficace.

Si $\Delta(\mathrm{d}\sigma/\mathrm{d}\Omega)$ est l'erreur faite sur la mesure de la section efficace de diffusion des muons, certainement beaucoup plus grande que celle des électrons, la différence de rayon quadratique qu'on peut espérer mesurer est:

$$|\langle r_0^2\rangle_\mu - \langle r_0^2\rangle_e| > \frac{3}{q^2}\left|\frac{\Delta(\mathrm{d}\sigma/\mathrm{d}\Omega)_\mu}{(\mathrm{d}\sigma/\mathrm{d}\Omega)_\mu}\right| .$$

Les formules précédentes se généralisent au cas de la diffusion sur des noyaux complexes, dans la mesure où l'application de l'approximation de Born se justifie. $F_P(q^2)$ est alors remplacé par le facteur de forme du noyau envisagé.

2. – Les effets des interactions anormales des muons dans leur diffusion à haute énergie sur des noyaux.

La section efficace de diffusion de Mott à haute énergie ($p \approx E$) sur *un noyau* est, respectivement, pour des interactions vectorielles ou scalaires:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_V = \frac{4E^2\cos^2\theta/2}{1 + (2E/M)\sin^2\theta/2}\left[\frac{\alpha Z}{2^2E^2\sin^2\theta/2} + \gamma_V h\right]^2 F^2(q^2) ,$$

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_S = \frac{1}{1 + (2E/M)\sin^2\theta/2}\left[\frac{Z^2\alpha^2\cos^2\theta/2}{4E^2\sin^4\theta/2} + 4h^2\gamma_S^2\sin^2\frac{\theta}{2} + \frac{2\alpha Z\gamma h m\mu}{E\sin^2\theta/2}\right] F^2(q^2) ,$$

où $\alpha = e^2/4\pi = 1/137$; γ_V et γ_S sont les constantes $g_S/4\pi$ et $g_V/4\pi$ définies à la Sec. 1·1. M est la masse du noyau, E l'énergie du muon incident et θ son angle de diffusion. h a été défini à la Sec. 1·1. $F(q^2)$, facteur de forme du noyau est pratiquement le même pour l'interaction normale et anormale.

En raison de l'importante diminution de la section efficace produite par le facteur de structure aux grands angles, une interaction anormale scalaire serait très difficile à mettre en évidence, alors que l'interaction vectorielle se mani-

festerait plus facilement. Les situations seraient intermédiaires pour des interactions pseudovectorielle ou tensorielle.

Pour calculer la section efficace de diffusion sur *des protons*, nous devons faire des hypothèses concernant les divers facteurs de structure vectoriel et tensoriel de l'interaction anormale. Sous l'hypothèse de l'existence d'un couplage nu fondamentale nucléon-boson vectoriel, il est facile de voir en théorie de perturbation que les facteurs de structure sont les mêmes que les facteurs électromagnétiques à part éventuellement une différence de couplage entre γ-pion et boson-pion. Si l'on admet de plus que les pions peuvent être considérés comme un état lié de paires de nucléons, ces couplages sont alors les mêmes.

Le facteur de structure de l'interaction anormale muon-proton est alors le produit des facteurs de structure du proton, par le facteur d'interaction, par échange du boson ($F(q^2) \sim 1/(q^2 + M_V^2)$) dont la variation est négligeable devant celle des facteurs de structure du proton ($M_V \gg 2m_\pi$).

On a alors pour la diffusion élastique:

$$\left(\frac{d\sigma}{d\Omega}\right)_P = \frac{4E^2 \cos^2 \theta/2}{1 + (2E/M) \sin^2 \theta/2} \left[\frac{\alpha}{4E^2 \sin^2 \theta/2} + \gamma_V\right]^2 \cdot \left[F_1^2 + \frac{q^2}{4M^2}\left[2(F_1 + xF_2)^2 \operatorname{tg}^2 \frac{\theta}{2} + x^2 F_2^2\right]\right],$$

F_1 et F_2 étant les facteurs de structure électrique et magnétique du proton, $x = 1.79$ le moment magnétique anormal du proton.

Nous allons maintenant étudier dans quelle mesure des expériences de diffusion des muons par les noyaux peuvent permettre de diminuer la borne de l'interaction anormale obtenue par l'étude des atomes mésiques.

Nous n'examinerons ici que le cas vectoriel, expérimentalement le plus favorable.

Pour espérer mettre en évidence une interaction anormale il faut d'une part que l'effet ne soit pas si faible qu'on ne puisse l'observer et, d'autre part, qu'il ne soit pas masqué par l'interaction électromagnétique. Dans le cas de la diffusion sur l'hydrogène, on a pour la différence de section efficace par stéradian σ_V qu'introduirait une interaction anormale

$$\sigma_V \simeq \frac{F_1^2(q^2)}{1 + (2E/M) \sin^2 \theta/2} \left(2\gamma\alpha \operatorname{cotg}^2 \frac{\theta}{2} + 4\gamma^2 E^2 \cos^2 \frac{\theta}{2}\right) \cdot \left[1 + \frac{q^2}{4M^2}\left(2(1+x)^2 \operatorname{tg}^2 \frac{\theta}{2} + x^2\right)\right],$$

et pour le rapport σ_V/σ_E, σ_E étant la section efficace de diffusion électromagnétique:

$$\left(\frac{\sigma_V}{\sigma_E}\right)_P \simeq 8\alpha^{-1}\gamma E^2 \sin^2 \frac{\theta}{2} + 16\alpha^{-2}\gamma^2 E^4 \sin^4 \frac{\theta}{2}.$$

Nous avons tracé sur la figure σ_V/σ_E en fonction de $2E \sin\theta/2$ pour diverses valeurs de γ. Nous avons pris $\gamma = +1.5 \cdot 10^{-7}\,(1/m_e^2)$ valeur correspondant à l'effet anormal observé (*) dans le noyau P, et $\gamma = \pm 3 \cdot 10^{-8}\,1/m_e^2$, $\gamma = \pm 0.6 \cdot 10^{-8}\,(1/m_e^2)$ et $\gamma = \pm 10^{-9}\,(1/m_e^2)$ valeurs de constante d'interaction qu'on est loin de pouvoir observer dans les atomes mésiques. Dans ce système d'unités, la constante des interactions faibles est $3 \cdot 10^{-13}\,(1/m_e^2)$.

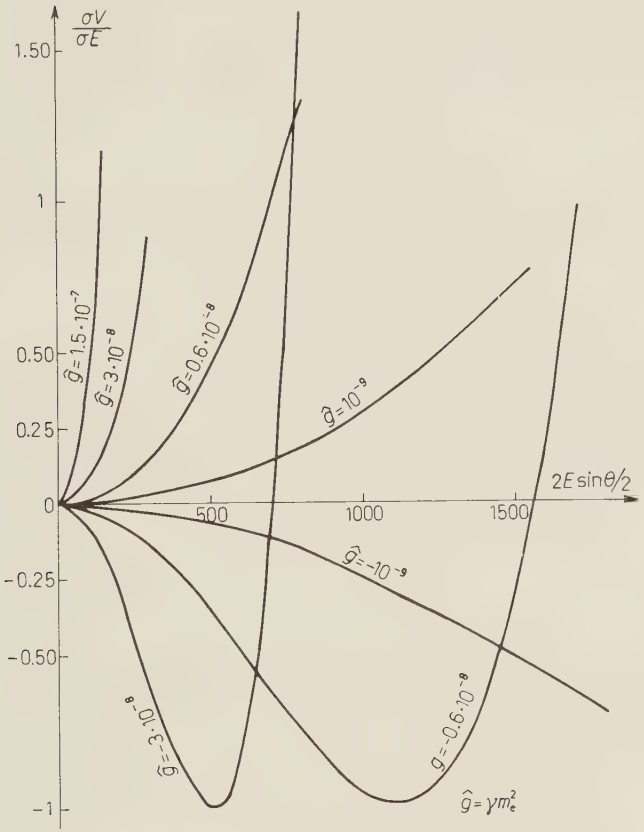


Fig. 1.

Pour détecter un effet anormal correspondant à une certaine valeur de γ, il faut donc choisir des conditions pour lesquelles $|\sigma_V/\sigma_E|$ soit plus grand ou égal à une valeur dépendant de la précision des mesures. On peut alors déterminer pour chaque énergie incidente les intervales angulaires favorables et évaluer l'ordre de grandeur de la section efficace anormale qu'il faudra mesurer

(*) Voir note (*) p. 705.

dans chaque cas:

$$\Delta\sigma = \int_{\theta_1}^{\theta_2} \sigma_V \Omega(\theta)\, \mathrm{d}\theta\,,$$

où $\Omega(\theta)\,\mathrm{d}\theta$, est l'angle solide égal à $2\pi \sin\theta\,\mathrm{d}\theta$ si les conditions expérimentales permettent l'utilisation des détecteurs annulaires. Pour les valeurs négatives de γ, il peut y avoir deux intervalles angulaires favorables correspondant à une diminution et à une augmentation de la section efficace de diffusion électromagnétique. Nous avons évalué $\Delta\sigma$ pour $|\sigma_V/\sigma_E| \geqslant 0.25$ et différentes valeurs de γ et de P et avons porté les résultats dans le Tableau II.

Tableau II.

	$p=300$ MeV/c	$p=1\,000$ MeV/c	$p=3\,000$ MeV/c	$p=10\,000$ MeV/c
$\gamma m^2 = 1.5\cdot 10^{-7}$	$90^\circ > \theta > 15^\circ$ $\Delta\sigma \sim 3.5\cdot 10^{-29}$			
$\gamma m^2 = 3\cdot 10^{-8}$	$90^\circ > \theta > 33^\circ$ $\Delta\sigma \sim 3\cdot 10^{-30}$	$30^\circ > \theta > 10^\circ$ $\Delta\sigma \sim 3\cdot 10^{-30}$	$30^\circ > \theta > 3^\circ$ $\Delta\sigma \sim 10^{-29}$	$10^\circ > \theta > 1^\circ$ $\Delta\sigma \sim 10^{-29}$
$\gamma m^2 = 0.6\cdot 10^{-8}$	$120^\circ > \theta > 79^\circ$ $\Delta\sigma \sim 10^{-30}$	$60^\circ > \theta > 22^\circ$ $\Delta\sigma \sim 0.3\cdot 10^{-30}$	$30^\circ > \theta > 8^\circ$ $\Delta\sigma \sim 0.5\cdot 10^{-30}$	$10^\circ > \theta > 2^\circ$ $\Delta\sigma \sim 0.5\cdot 10^{-30}$
$\gamma m^2 = 10^{-9}$			$30^\circ > \theta > 18^\circ$ $\Delta\sigma \sim 10^{-32}$	$10^\circ > \theta > 5^\circ$ $\Delta\sigma \sim 10^{-32}$

Les différences de section efficace $\Delta\sigma$ sont données en cm^2 et les angles θ_1 et θ_2 en degrés θ_1 est déterminé par la condition $|\sigma_V/\sigma_E| > 0.25$ et la décroissance de F^2 permet de choisir θ_2 de façon que la différence de section efficace pour des angles $> \theta_2$ soit négligeable.

Dans tous les cas nous avons pris $\Delta\Omega = 2\pi \sin\theta\,\mathrm{d}\theta$.

Nous donnons ici à titre indicatif les valeurs de $\Delta\sigma$ pour $E = 10$ GeV. Si notre formule s'applique encore à cette énergie, on voit en effet que $\Delta\sigma$ est la même qu'à 3 GeV.

Pour $P = 300$ MeV/c le valeurs données ont été établies à l'aide des formules données plus haut, corrigées pour tenir compte de $P \neq E$.

Dans le cas de noyau (A, Z), on a:

$$\frac{\sigma_V}{\sigma_E} = 8\alpha^{-1}\gamma \frac{h}{Z} E^2 \sin^2\frac{\theta}{2} + 16\alpha^{-2}\gamma^2 \left(\frac{h}{Z}\right)^2 E^4 \sin^4\frac{\theta}{2}$$

(h a été défini à la Sec. 1·1). Pour des conditions identiques à celles envisagées dans le cas du proton, on a un gain appréciable sur $(\sigma_V)_H$ si $F^2(q^2)Zh > 1$. Mais $F^2(q^2)$ décroît très rapidement lorsqu'on augmente le transfert d'impul-

sion (pour un q de 306 MeV/c, $F^2(q^2)ZA$ vaut environ 0.7 pour le carbone); ainsi même en dehors des difficultés supplémentaires provoquées par les diffusions inélastiques lorsqu'on utilise des noyaux autres que le proton, leur intérêt diminue beaucoup pour des $q > 200$ MeV/c. Toutefois, la diffusion semi-élastique par les nucléons à l'intérieur du noyau devient alors plus importante que la diffusion élastique et il peut y avoir un intérêt expérimental à l'étudier pour la recherche d'une interaction anormale muon-nucléon.

* * *

Nous remercions M.lle Colette Petit pour son aide dans les calculs numériques.

RIASSUNTO (*)

Si determinano a partire dai dati sperimentali recenti (atomi mesici, momento magnetico) dei limiti per le masse e le costanti d'accoppiamento dei bosoni, scalari, vettoriali e pseudoscalare, adatti a spiegare la massa anomala del muone. Si discutono in seguito le condizioni per mettere in evidenza una interazione anomala muone-nucleone nella diffusione elastica sui nuclei.

(*) *Traduzione a cura della Redazione.*

An Unified Theory of Direct Reactions.

A. Agodi (*) and E. Eberle

Istituto di Fisica dell'Università - Catania
Centro Siciliano di Fisica Nucleare - Catania

(ricevuto il 26 Luglio 1960)

Summary. — A new dispersion theory of nuclear reactions is presented which is a generalization of the existing ones. The theory is centered around the possibility of defining a class of problems equivalent to the original scattering problem in the sense that their solutions all determine the same transition matrix elements. These equivalent problems are generated by suitably projecting the original one on a subspace of the linear space spanned by the continuum eigenstates of the total Hamiltonian, containing the initial and final channel states. A derivation of the Breit-Wigner formula analogous to Feshbach's can be given in our theory, independently of any configuration space representation. The direct processes are then defined by performing the energy average on the transition matrix elements, so that the contributions from the long-lived « compound » states are lost. In the case of nucleon-nucleus scattering this is shown to give an alternative definition of the optical model quite analogous to the conventional ones though perhaps more accessible to a field theoretical analysis. Moreover it is shown that the well known surface contributions to direct reactions can be isolated quite naturally in our formalism from the other direct contributions.

---

1. – In the present paper a formulation of the theory of nuclear reactions is given, developped out of generally well known discussions [1] on the physical definition of direct reactions.

(*) This work has been supported in part by a grant of « Fondazione A. Della Riccia » during this author's stay at C.E.N., Saclay, France, and in part by C.R.R.N., Palermo.

(1) V. F. Weisskopf: *Physica*, **22**, 952 (1956); D. H. Wilkinson: *Physica*, **22**, 1039 (1956); C. T. De Dominicis: *Compt. Rend. Congrés Intern. Physique Nucl.* (Paris, 1958), p. 87; R. E. Peierls: *Lectures at the Summer Institute for Theor. Phys.* (University of Colorado, Boulder, 1958).

It seems that such a formulation provides a suitable framework both for describing the results of our investigations on the surface components of photonuclear processes (2) as well as for studying the relations existing among the direct components of different nuclear reactions.

It is quite well known that a dispersion theory can be used for analysing direct nucleon-nucleus interactions (3). And in so far as the other direct interactions can be explained in terms of this one, it is also clear that on the basis of such a theory one can obtain an unified understading (4) of these direct interactions. In fact we used at the very beginning of our work (2,5) the general dispersion theory of nuclear reactions proposed by C. BLOCH (3), which can be specialized to give either that of WIGNER and EISENBUD (6) or that of PEIERLS and KAPUR (7).

A common feature of these dispersion theories is the separation of configuration space into two regions, being the « interaction-region » and the « channel-region ». A basic set of states is defined which is complete in the interaction region and is physically related to a set of decaying states of the compound system. A second set of states, which is complete in the channel region, is also introduced, and is designed to describe the information about groups of particles (elementary or composite) without mutual interactions, so as they are either prepared or revealed with suitably chosen physical devices.

The general scattering state is then analyzed by means of the two complete sets.

The wide applicability and power of this sort of theories are well known

It has however been pointed out (8) that they also have some undesirable features: let us mention here *e.g.* the explicit dependence of the transition amplitudes on the channel radii, the difficulty of taking into account the effects of the exclusion principle and of hard core nucleon-nucleon potentials, together with less apparent difficulties connected *e.g.* with the complications inherent in trying a field theoretical analysis of a multichannel theory defined in con-

---

(2) A. AGODI, E. EBERLE and L. SERTORIO: *Nuovo Cimento*, **13**, 1279 (1959).

(3) R. G. THOMAS: *Phys. Rev.*, **97**, 224 (1955); **100**, 25 (1955); C. BLOCH: *Nucl. Phys.*, **4**, 503 (1957); T. TEICHMAN and E. P. WIGNER: *Phys. Rev.*, **87**, 123 (1952); A. M. LANE and R. G. THOMAS: *Rev. Mod. Phys.*, **30**, 257 (1958); G. E. BROWN and C. T. DE DOMINICIS: *Proc. Phys. Soc.*, A **70**, 668 (1957).

(4) G. E. BROWN and C. T. DE DOMINICIS, *Proc. Phys Soc.*, A **70**, 686 (1957); G. E. BROWN: *Rev. Mod. Phys.*, **31**, 893 (1959).

(5) A. AGODI: *Suppl. Nuovo Cimento*, **11**, 431 (1959).

(6) E. P. WIGNER and L. EISENBUD: *Phys. Rev.*, **72**, 29 (1947).

(7) P. I. KAPUR and R. E. PEIERLS: *Proc. Roy. Soc.*, A **166**, 277 (1937).

(8) H. FESHBACH: *Ann. Phys.*, **5**, 357 (1958); see also: A. M. LANE: *Nucl. Phys.*, **11**, 625 (1959); G. E. BROWN quoted under (4) and A. M. LANE and R. G. THOMAS quoted under (3).

figuration space ([9]) and with the arbitrariness yet existing in the formal definition and in the physical meaning of the decaying states ([10]).

A dispersion formalism has been proposed by FESHBACH ([11]), which does not introduce the separation of configuration space into two regions and so has no explicit dependence on channel radii. Still it has in common with the previous theories the other mentioned undesirable features. As explicitly noted by FESHBACH the entire discussion of his paper ([11]) might be easily rewritten in terms of the projection operator formalism employed by himself in an earlier review article ([12]) on the optical model.

Bloch's theory also can be given in terms of a similar formalism *e.g.* by using as a basic projector that one selecting out those components of the general state lying in the interaction region.

The starting point of the present theory might be the remark that if a projection operator can be defined, having a well established physical meaning then one can correspondingly give a meaning to a problem equivalent to the original scattering problem in the subspace selected out by means of that projection operator. It turns out that in this way one is naturally led to the construction of a dispersion theory.

Now, in the general scattering problem the whole of physical information is obtained from measurements on non interacting (elementary or composite) particles, *i.e.* from observations on channel states.

The eigenstates of the total hamiltonian (belonging to the continuous spectrum) also are physically determined (and in fact labelled) from their lying asymptotically in the subspace spanned by some specified set of channel states.

Moreover we point out that the general scattering problem essentially refers to two channels: so we expect that an equivalent problem, in a subspace constructed out of just the two relevant sets of channel states can be defined, whose solution determines the same scattering amplitude as that of the original scattering problem.

The investigation of this equivalent problem when the original one is that of nucleon-nucleus scattering, leads, with a suitable choice of the channel states, to the optical model analysis of the scattering.

It is well known ([13]) that such an analysis is essentially on those compo-

---

([9]) H. EKSTEIN: *Phys. Rev.*, **101**, 880 (1956); J. M. JAUCH: *Helv. Phys. Acta*, **31**, 127 (1958).

([10]) R. E. PEIERLS: *Proc. Glasgow Conf. on Nucl. and Meson Physics* (1954), p. 296; P. J. MATTHEWS and A. SALAM: *Phys. Rev.*, **112**, 283 (1958); B. ZUMINO: New York University Res. Rep. CX-23 (1956), unpublished; J. SCHWINGER: *Ann. Phys.*, **9**, 169 (1960).

([11]) H. FESHBACH: quoted under ([8]).

([12]) H. FESHBACH: *Ann. Rev. Nucl. Sci.*, **8**, 49 (1958).

([13]) F. L. FRIEDMAN and V. F. WEISSKOPF: *Niels Bohr and the Development of Physics* (London, 1955), p. 134; C. T. DE DOMINICIS: *Journ. Phys. Rad.*, **19**, 1 (1958).

nents of the nucleon-nucleus elastic scattering which are isolated by doing experiments in conditions of poor energy resolution.

These conditions can be included in the theory as a prescription on the « admissible » incoming or outgoing waves in the relevant channel. These conditions also are found ([13]) to select out of the experimental cross-section that part of it due to « direct processes » *i.e.* to processes occurring without compound nucleus formation.

The study of such « direct processes » provides a powerful method for investigating the structure of nuclei: it is well known that when a direct contribution can be isolated in some nuclear reactions (*e.g.* pick-up or photonuclear reactions) then one is led to regard it as characterized ([14]) by a residual nucleus (in the final channel) left in a state of low excitation or perhaps in the ground state (the latter being a particularly meaningful statement in the case of light nuclei).

Here we have again a prescription for selecting « admissible states » in the final channel.

It seems that such prescriptions can be introduced in the formal theory by defining projection operators such that the « admissible states » belong to their eigenvalue 1.

Since the selection of « admissible states » only can be physically realized by means of a choice of preparing or detecting devices it seems quite plausible to give a definition of the corresponding projection operators by means of equations involving channel states only. The use of projectors defined in this way is one of the essential features of the theory we are presenting here.

We now summarize the content of the paper. In Section **2**, we define the formal basis of the theory and construct the general equivalent problem; in Section **3** we define the elimination of the closed channels and thus introduce the unstable compound states; in Section **4** we perform the energy averaging and the reduction of the problem to the subspace of the chosen channel states ; Section **5** is devoted to a discussion of the different contributions to the average transition amplitude and Section **6** to just stating some remarks on the optical model nucleon-nucleus scattering, and on surface direct effects in nuclear reactions whose detailed analysis will be given elsewhere.

2. – Let $|\alpha Ea\rangle$ be the general element of a basic set of channel states, complete in channel α. The labels are the values of some arbitrary complete set of commuting observables including the energy.

([14]) S. T. Butler: *Phys. Rev.*, **106**, 272 (1957); P. Gugelot: *Phys. Rev.*, **93**, 424 (1954); G. E. Brown and J. S. Levinger: *Proc. Phys. Soc.*, A **71**, 733 (1958); B. L. Cohen and A. G. Rubin: *Phys. Rev.*, **114**, 1143 (1959).

We define the channel hamiltonian H_α so that, for each α

$$(E - H_\alpha)|\alpha Ea\rangle = 0 \,. \qquad \text{(all } a\text{)}. \tag{1}$$

It is assumed that H_α has a purely continuous spectrum ([15]) including all real values of $E \geqslant E_\alpha$. We will refer in the following to E_α as the threshold energy of channel α.

Also we introduce for each α two sets of states $|\alpha Ea \pm\rangle$, being eigenstates of the total hamiltonian belonging to the eigenvalue E of its continuous spectrum:

$$(E - H)|\alpha Ea \pm\rangle = 0 \,. \tag{2}$$

By definition $\exp[-iHt]|\alpha Ea \pm\rangle$ behaves like $\exp[-iH_\alpha t]|\alpha Ea\rangle$ as t goes to $\pm\infty$ respectively; we write this in the form

$$\lim_{t\to\pm\infty} \exp[iH_\alpha t] \exp[-iHt]|\alpha Ea \pm\rangle = |\alpha Ea\rangle \,. \tag{3}$$

This amounts, in the usual way, to put

$$|\alpha Ea \pm\rangle = |\alpha Ea\rangle + \frac{1}{E^{(\mp)} - H_\alpha} V_\alpha |\alpha Ea \pm\rangle = \left[1 + \frac{1}{E^{(\mp)} - H}\right] V_\alpha |\alpha Ea\rangle \,, \tag{4}$$

with $V_\alpha = H - H_\alpha$ and with $E^{(\pm)} = E \pm i\varepsilon$, all expressions containing it being evaluated in the limit when ε is made to vanish through positive values.

We do not discuss here the precise mathematical meaning of (3): we just mention that the Hilbert space embedding Jauch's ([15]) multichannel scattering system is assumed here to be a subspace of the linear space commonly used in the formal theory of collisions and reaction processes ([16]).

More precisely we understand our states $|\alpha Ea\rangle$ and $|\alpha Ea \pm\rangle$ as determining a representation in the subspaces belonging to the eigenvalue 1 of Jauch's projectors E_α, $F_\mp^{(\alpha)}$, respectively.

The $|\alpha Ea +\rangle$ ($|\alpha Ea -\rangle$) clearly correspond to the well known incoming (outgoing) wave solutions of (2).

The transition probability amplitudes from the state αEa to the state $\alpha' E' a'$ can be defined, essentially in the usual way ([15,16]), as the scalar products $\langle \alpha' E' a' + | \alpha Ea - \rangle$.

([15]) J. M. JAUCH: *Helv. Phys. Acta*, **31**, 661 (1958).

([16]) B. LIPPMANN and J. SCHWINGER: *Phys. Rev.*, **79**, 469 (1950); M. GELL-MANN and M. L. GOLDBERGER: *Phys. Rev.*, **91**, 398 (1953); H. E. MOSES: *Nuovo Cimento*, **1**, 103 (1955); B. ZUMINO quoted under ([10]).

Use of (4) and of some straightforward algebra gives the hardly new equation

$$\langle \alpha' E' a' + | \alpha E a - \rangle = \delta_{\alpha\alpha'} \langle \alpha' E' a' | \alpha E a \rangle - 2\pi i \, \delta (E - E') \cdot \cdot \langle \alpha' E' a' + | V_\alpha | \alpha E a \rangle \,, \tag{5}$$

in terms of which one calculates cross-sections in the standard way ([16,17]). Equations (1) to (5) are the formal basis of the following developments.

First of all we now point out that the r.h.s. of eq. (5) can be exhibited in terms of matrix elements of operators in any subspace containing the channel states belonging to the initial and to the final channel.

Let $\mathscr{P}$ be the projector selecting out from any state $|\psi\rangle$ its components lying in that subspace. For the moment being we leave it undetermined up to the condition that for all values of Ea and for the two α and α' channels indicated in (5)

$$(\mathscr{P} - 1) | \alpha E a \rangle = (\mathscr{P} - 1) | \alpha' E a \rangle = 0 \,. \tag{6}$$

For the general state $|\psi\rangle$ and the general operator T we introduce the notations:

$$(\mathscr{P} + \mathscr{Q}) | \psi \rangle = | \psi \rangle \,; \quad \mathscr{P} | \psi \rangle = | i\psi \rangle \,; \quad \mathscr{Q} | \psi \rangle = | e\psi \rangle \,;$$

$$\mathscr{P} T \mathscr{P} = T_i \,; \quad \mathscr{Q} T \mathscr{Q} = T_e \,; \quad \mathscr{Q} T \mathscr{P} = T_{ei} \,; \quad \mathscr{P} T \mathscr{Q} = T_{ie} \,,$$

i, e are for «included» and «excluded» respectively, referring to the choice operated by $\mathscr{P}$ on the elements of our linear space. We wish to show that

$$\langle \alpha' E' a' + | \alpha E a - \rangle = \delta_{\alpha\alpha'} \langle i\alpha' E' a' | i\alpha E a \rangle - \\ - 2\pi i \, \delta (E - E') \langle i\alpha' E' \alpha' + | \left[V_{\alpha i} + \mathscr{W}_i (E'^{(+)}) \right] | i\alpha E a \rangle \tag{7}$$

holds true, with the definition

$$\mathscr{W}_i(z) = H_{ie} \frac{1}{z - H_e} H_{ei} \,. \tag{8}$$

In orther to do this we point out that, owing to (6) one has

$$\langle \alpha' E' a' | \alpha E a \rangle = \langle i\alpha' E' a' | i\alpha E a \rangle$$

([17]) J. M. Jauch quoted under ([9]).

and that from eq. (4) and the operator identity

$$\frac{1}{A} = \frac{1}{B}\left[1 + (B - A)\frac{1}{A}\right], \tag{9}$$

one also obtains

$$\begin{aligned} |i\alpha' E' a' \pm\rangle &= |i\alpha' E' a'\rangle + \mathscr{P}\frac{1}{E'^{(\mp)} - H} V_{\alpha'} |\alpha' E' a'\rangle = \\ &= |i\alpha' E' a'\rangle + \mathscr{P}\frac{1}{E'^{(\mp)} - H_i}\left[1 + (H - H_i)\frac{1}{E'^{(\mp)} - H}\right] V_{\alpha'} |\alpha' E' a'\rangle = \\ &= |i\alpha' E' a'\rangle + \frac{1}{E'^{(\mp)} - H_i} V_{\alpha i} |i\alpha' E' a'\rangle + \frac{1}{E'^{(\mp)} - H_i} H_{ie} |e\alpha' E' a' \pm\rangle . \end{aligned} \tag{10}$$

With a quite similar procedure, taking into account that eq. (6) can be written $\mathscr{Q}|\alpha' E' a'\rangle = 0$ we find

$$|e\alpha' E' a' \pm\rangle = \frac{1}{E'^{(\mp)} - H_e} H_{ei} |i\alpha' E' a' \pm\rangle , \tag{11}$$

owing to

$$\mathscr{Q} V_{\alpha'} |\alpha' E' a'\rangle = H_{ei} |i\alpha' E' a'\rangle . \tag{12}$$

Substitution of (11) into (10) gives

$$\begin{aligned} |i\alpha' E' a' \pm\rangle = &\left[1 + \frac{1}{E'^{(\mp)} - H_i} V_{\alpha' i}\right] |i\alpha' E' a'\rangle + \\ &+ \frac{1}{E'^{(\mp)} - H_i} \mathscr{W}_i(E'^{(\mp)}) |i\alpha' E' a' \pm\rangle . \end{aligned} \tag{13}$$

Always remembering (6) we have, from (11), (12), that

$$\begin{aligned} \langle \alpha' E' a' + | V_\alpha | \alpha E a\rangle &= \langle i\alpha' E' a' + | V_{\alpha i} | i\alpha E a\rangle + \langle e\alpha' E' a' + | H_{ei} | i\alpha E a\rangle = \\ &= \langle i\alpha' E' a' + | [V_{\alpha i} + \mathscr{W}_i(E'^{(+)})] | i\alpha E a\rangle \end{aligned}$$

holds true and thus also eq. (7).

In order to evaluate the transition probability amplitude one could either solve the integral equation being the first of (4) and then use eq. (5) or, alternatively, solve an equivalent problem being the integral equation (13) in the subspace belonging to the eigenvalue 1 of $\mathscr{P}$ and then use eq. (7).

Clearly the equivalent problem is defined only when $\mathscr{P}$ is completely determined and if $\mathscr{W}_i(E^{(\mp)})$ can be constructed explicitly or otherwise regarded

as known: what we have described thus far is the general one of a broad class of problems equivalent to the original scattering problem.

The theory of direct processes we are presenting here is essentially based on a suitable choice of $\mathscr{P}$ which seems to have a clear physical meaning and also leads to a dispersion formalism not having the undesirable features we mentioned in the introduction as common to all existing dispersion theories.

3. – As it appears from eq. (8) the matrix elements of $\mathscr{W}_i$ contributing to (7) describe transitions from the initial to the final state through formation and decay of some intermediate states lying in the excluded states subspace. This suggests that the choice of $\mathscr{P}$ can be related to the determination of the unstable states describing the so called compound system stage of the general nuclear reaction according to Weisskopf's [1] picture.

A number of more or less equivalent definitions of unstable states have been studied in recent years [10]: a common assumption is that such unstable states have no asymptotic limit whatever *i.e.* do not allow a channel state characterization.

Thus it seems quite reasonable to impose on our compound system decaying states the essential condition of being orthogonal to some set of channel states. We are maintaining in this way that, whenever the physical information on the system under observation cannot be completely described by means of a state being a superposition of that set of channel states, then we must introduce some new states. They do not refer, in agreement with the identification here proposed, to some analysis of the general state in terms of measurements on groups of particles either before or after their mutual interactions: they just are needed to describe information on the general state in so far as it refers to relations amongst two sets of measurements on non interacting groups of particles performed the one before and the second after their mutual interaction.

In this way we are allowed to indicate shortly the unstable states as describing just the structure of the interactions.

We now remark that, working at any given energy E, only a subset of all channel states can be actually observed, *i.e.* those belonging to the channels whose threshold energy is not greater than E. They can be used to define the subspace of the channels open at energy E.

We have found convenient, as it will be apparent from the following developments, to introduce the subspace of the channels open in the energy interval

$$[E_1; \Delta_1] = E_1 - \frac{\Delta_1}{2} \leqslant E \leqslant E_1 + \frac{\Delta_1}{2}.$$

In order to define it precisely let us introduce the characteristic function

$\chi_{E_1 \Delta_1}(E)$ of the interval

$$\chi_{E_1 \Delta_1}(E) = \int_{E_1 - (\Delta_1/2)}^{E_1 + (\Delta_1/2)} \delta(E' - E)\, \mathrm{d}E' .$$

Also we define the characteristic function of the interval on channel α:

$$\chi^{(\alpha)}_{E_1 \Delta_1}(E) = \int_{E_1 - (\Delta_1/2)}^{E_1 + (\Delta_1/2)} \delta(E' - E)\, \mathrm{d}E' \int_{E_\alpha}^{\infty} \delta(E'' - E)\, \mathrm{d}E'' .$$

We then define the open channels subspace of the energy interval $[E_1; \Delta_1]$ as constructed by completing in the standard way ([18]) the linear manifold spanned by all superpositions of the states $|\alpha E a\rangle \chi^{(\alpha)}_{E_1 \Delta_1}(E)$. The projection operator on that subspace will be denoted $P_{E_1 \Delta_1}$.

Clearly the matrix elements (5) only have a physical meaning for transitions between open channels. So for any value of $E (= E')$ in the energy interval $[E_1; \Delta_1]$ the states in the initial and in the final channel to which (5) refers satisfy eq. (6) when $\mathscr{P}$ is substituted by $P_{E_1 \Delta_1}$.

Therefore all equations of the previous section hold true after having done such a substitution. The new equations, with $P_{E_1 \Delta_1}$ in the place of $\mathscr{P}$, will be referred to in the following by just changing the index « i » with « o » and « e » with « c »: « o » and » c » are for « open » and « closed » channel respectively, the energy interval $[E_1; \Delta_1]$ being understood.

Owing to non existence of real transitions from the open to the closed channels we remark that it must be

$$\mathscr{W}^{\dagger}_{o}(E^{(\pm)}) = \mathscr{W}_{o}(E^{(\pm)}) \qquad i.e. \qquad \mathscr{W}_{o}(E^{(\pm)}) = \mathscr{W}_{o}(E) .$$

Since H_c, the reduction of the total hamiltonian to the subspace of the closed channels, will in general have a mixed spectrum, we will introduce its spectral resolution into (8) as

$$\mathscr{W}_{o}(E) = \sum_n H_{oc} |cn\rangle \frac{1}{E - \mathscr{E}_n} \langle cn| H_{co} + \int \mathrm{d}\mathscr{E}\, H_{oc} |c\mathscr{E}\rangle \frac{1}{E - \mathscr{E}} \langle c\mathscr{E}| H_{co} ,$$

with obvious meaning of the symbols.

In order to clarify the connection of the $|cn\rangle$, $|c\mathscr{E}\rangle$ states with the un-

---

([18]) N. I. ACHIESER and I. M. GLASMANN: *Theorie der linearen Operatoren im Hilbert Raum* (Berlin, 1954); F. RIESZ and B. DE SZ. NAGY: *Leçons d'analyse fonctionelle* (Budapest, 1952).

stable compound states of our system, we will just indicate some results one can easily obtain, following formally Feshbach's procedure ([8]).

We will refer to the two especially interesting cases when either one or a great number of eigenvalues $\mathscr{E}_n$, of the point spectrum of H_c, are lying in the chosen energy interval $[E_1; \Delta_1]$. In both cases $\mathscr{W}_o(E)$ can be split into two parts: one of them, $\mathscr{V}$, containing the contributions from the eigenvalues lying in the chosen interval, is rapidly varying with E, the other, $\mathscr{U}$, is not,

$$\mathscr{W}_o(E) = \mathscr{V} + \mathscr{U}. \tag{14}$$

We wish to show that we can separate out in the transition amplitude the contributions due to $\mathscr{V}$, rapidly fluctuating as the energy is varying in the chosen energy interval.

After substitution of $\mathscr{P}$ with $P_{E_1\Delta_1}$ and remembering eq. (14), eq. (7) reads

$$\langle \alpha' E' a' + | \alpha E a - \rangle = \delta_{\alpha\alpha'} \langle o\alpha' E' a' + | o\alpha E a \rangle - \\ - 2\pi i\, \delta(E - E') \langle o\alpha' E' a' + | [V_{\alpha o} + \mathscr{V} + \mathscr{U}] | o\alpha E a \rangle . \tag{7'}$$

Some straightforward manipulations, by using eq. (9) once more, allow us to transform (13) into

$$| o\alpha' E' a' \pm \rangle = \left[1 + \frac{1}{E'^{(\mp)} - H_o - \mathscr{U}} (V_{\alpha' o} + \mathscr{U}) \right] | o\alpha' E' a' \rangle + \\ + \frac{1}{E'^{(\mp)} - H_o - \mathscr{U}} \mathscr{V} | o\alpha' E' a' \pm \rangle = | \psi^{(\pm)}_{\alpha' E' a'} \rangle + \frac{1}{E'^{(\mp)} - K} \mathscr{V} | o\alpha' E' a' \pm \rangle , \tag{13'}$$

with the definitions:

$$| \psi^{(\pm)}_{\alpha' E' a'} \rangle = \left[1 + \frac{1}{E'^{(\mp)} - K} \mathscr{U}_{\alpha'} \right] | o\alpha' E' a' \rangle , \tag{15a}$$

$$K = H_o + \mathscr{U}; \qquad \mathscr{U}_{\alpha'} = V_{\alpha' o} + \mathscr{U} . \tag{15b}$$

We have then, on the energy shell, $E = E'$:

$$\langle o\alpha' E a' + | [\mathscr{U}_\alpha + \mathscr{V}] | o\alpha E a \rangle = \langle o\alpha' E a' + | \mathscr{U}_\alpha | o\alpha E a \rangle + \\ + \langle o\alpha' E a' + | \mathscr{V} \left[| \psi^{(-)}_{\alpha E a} \rangle - \frac{1}{E^{(+)} - K} \mathscr{U}_\alpha | o\alpha E a \rangle \right] = \\ = \langle \psi^{(+)}_{\alpha' E a'} | \mathscr{U}_\alpha | o\alpha E a \rangle + \langle o\alpha' E a' + | \mathscr{V} | \psi^{(-)}_{\alpha E a} \rangle , \tag{16}$$

which is the previously mentioned separation of the rapidly varying contributions from the others.

It is easy now (and for this reason we do not give details here) to obtain by Feshbach's ([8]) procedure, in the case of a single eigenvalue $\mathscr{E}_n$ lying in the chosen energy range

$$\mathscr{V} = H_{oc}|cn\rangle \frac{\chi_{E_1 A_1}(\mathscr{E}_n)}{E - \mathscr{E}_n} \langle cn|H_{oc}\,,$$

$$\langle o\alpha' Ea'+|\mathscr{V}|\psi^{(-)}_{\alpha Ea}\rangle = \frac{\langle \psi^{(+)}_{\alpha' Ea'}|H_{oc}|cn\rangle\langle cn|H_{co}|\psi^{(-)}_{\alpha Ea}\rangle}{E - \mathscr{E}_n - \langle cn|H_{co}(1/(E^{(+)} - K))H_{oc}|cn\rangle} =$$

$$= \sum_{\alpha'' a''} \langle \psi^{(+)}_{a' Ea'}|\psi^{(-)}_{\alpha'' Ea''}\rangle \frac{\langle cn|H_{oc}|\psi^{(-)}_{\alpha'' Ea''}\rangle^* \langle cn|H_{co}|\psi^{(-)}_{\alpha Ea}\rangle}{E - \mathscr{E}_n - \mathrm{Re}\langle cn|H_{co}(1/(E^{(+)} - K))H_{oc}|cn\rangle + i\pi \sum_{\alpha'' a''} |\langle cn|H_{co}|\psi^{(-)}_{\alpha'' Ea''}\rangle|^2},$$

which is, apart from the generalized notations, the Breit-Wigner formula for an isolated resonance.

In the case of many levels $\mathscr{E}_n$ lying in the chosen energy range

$$\mathscr{V} = \sum_n H_{oc}|cn\rangle \frac{\chi_{E_1 A_1}(\mathscr{E}_n)}{E - \mathscr{E}_n} \langle cn|H_{co}\,.$$

Denoting with $X^{(\nu)} \equiv \{X^{(\nu)}_n\}$ the solution of

$$\sum_{n'} X^{(\nu)}_{n'} \left[(E_\nu - \mathscr{E}_{n'})\delta_{n'n} - \langle cn'|H_{co} \frac{1}{E^{(+)} - K} H_{oc}|cn\rangle \right] = 0$$

belonging to the (complex) eigenvalue E_ν, and writing $X^{(\nu)\dagger} \equiv \{X^{(\nu)\dagger}_{n'}\}$ the adjoin of $X^{(\nu)}$ satisfying

$$\sum_n X^{(\nu)\dagger}_n X^{(\nu')}_n = \delta_{\nu\nu'} \quad ; \quad \sum_\nu X^{(\nu)\dagger}_n X^{(\nu)}_{n'} = \delta_{nn'}\,,$$

it can be shown that

$$\langle o\alpha' Ea'+|\mathscr{V}|\psi^{(-)}_{\alpha Ea}\rangle = \sum_{nn'\nu} \langle \psi^{(+)}_{\alpha' Ea'}|H_{oc}|cn'\rangle \frac{X^{(\nu)\dagger}_{n'} X^{(\nu)}_n}{E - E_\nu} \langle cn|H_{co}|\psi^{(-)}_{\alpha Ea}\rangle\,. \tag{17}$$

Again, since E_ν is in general complex, this has the form of a sum over resonances.

The decaying states are related to the linear combinations $\sum_n |cn\rangle X^{(\nu)\dagger}_n$ of the eigenstates $|cn\rangle$ whose eigenvalues are lying in the energy interval $[E_1; A_1]$. It could be easily shown that for (21) the Breit-Wigner form holds on the average *i.e.* there are sum rules relating the denominators to the imaginary parts of the E_ν, allowing the well known interpretation of the giant resonances *e.g.* when applied to the case of nucleon-nucleus elastic scattering.

Since we are interested in a definition of direct processes we will not give here a more detailed analysis of these sum rules and of the properties of the decaying states.

We merely stress that the basic condition we have imposed to such states, of being orthogonal to the open channels subspace in the chosen energy interval, does not determine them completely: in fact it only determines the subspace spanned by them, though it is sufficient to introduce what we would like to call a «natural» dispersion formalism.

4. – In order to introduce our definition of direct processes we remember that, in the case of nucleon-nucleus elastic scattering, such processes have been studied by means of the so called «optical model», which is a theory of the average transition matrix, the average being on the energy.

Now it is straightforward to show that the argument leading to the relation between average transition matrix and undelayed or «instantaneous» interactions can be generalized.

In fact let us define in the open channels subspace, wave packets $|f_{\alpha a}(E\,\Delta)\rangle$:

$$f_{\alpha a}(E\,\Delta)\rangle = \int \mathrm{d}E'\, \chi^{(\alpha)}_{E'\,\Delta_1}(E')\,|\alpha E' a\rangle \qquad \chi_{E\,\Delta}(E) = \int \mathrm{d}E'\, \chi^{(\alpha)}_{E\,\Delta}(E')\,|\alpha E' a\rangle$$

for every channel α and for every $[E, \Delta] \subset [E_1; \Delta_1]$.

Let us then consider the state determined by an incoming wave in channel α corresponding to $|f_{\alpha a}(E\Delta)\rangle$.

At a sufficiently long time after its preparation we expect that outgoing waves will develop in all other open channels. The density matrix [19] of the state described by such outgoing waves in channel α' will be

$$(Ea\|\varrho_{\alpha'}\|E'a') = \int \mathrm{d}E_1 \int \mathrm{d}E_1' \langle \alpha' Ea + |\alpha E_1 a_1 - \cdot \cdot \chi^{(\alpha)}_{E\,\Delta}(E_1)\,\chi^{(\alpha)}_{E\,\Delta}(E_1')\langle \alpha E_1' a_1 - |\alpha' E' a' + \rangle \,. \tag{18}$$

At any time t after the matrix of eq. (18) can be regarded to describe the state in channel α', as prepared by means of the reaction under investigation, the response of a radiation detector will be represented [19,20] by an operator $F_{\alpha'}$.

The intensity to be measured can be represented *e.g.* by the mean value

$$\sum_{a a'} \int \mathrm{d}E \int \mathrm{d}E' \exp\left[-\,i(E - E')t\right](E'a'\|F\alpha' Ea)\,(Ea\|\varrho_{\alpha'}\|E'a') = \\ = \mathrm{Tr}\, F_{\alpha'}(t)\varrho_{\alpha'} = \mathrm{Tr}\, F_{\alpha'}\varrho_{\alpha'}(t)\,.$$

[19] U. Fano: *Rev. Mod. Phys.*, **29**, 76 (1957).

[20] U. Fano and G. Racah: *Irreducible Tensorial Sets* (New York, 1959).

Now the probability that if the system is in the state described by the matrix of eq. (18) then it is also in some specified state $|f_{\alpha'a'}(E\Delta)\rangle$ at any time between 0 and τ will be given by

$$(19)\qquad \int_0^{\tau} \mathrm{d}t \langle f_{\alpha'a'}|\varrho_{\alpha'}(t)|f_{\alpha'a'}\rangle = \int \mathrm{d}E \int \mathrm{d}E' \frac{i}{E-E'}\{\exp[-i(E-E')\tau]-1\}\cdot \cdot \chi_{E\Delta}^{(\alpha')}(E)\,\chi_{E\Delta}^{(\alpha')}(E')(Ea'\|\varrho_{\alpha'}\|E'a')\,.$$

Two limiting cases are especially interesting, *i.e.* those obtained for $\tau \to +\infty$ (practically $\tau \gg 1/\Delta$) and for $\tau \to +0$ (practically $\tau \ll 1/\Delta$).

It is straightforward to show that one has

$$(20)\qquad \int_0^{\tau \gg (1/\Delta)} \mathrm{d}t \langle f_{\alpha'a'}|\varrho_{\alpha'}(t)|f_{\alpha'a'}\rangle \approx \int_0^{\infty} \langle f_{\alpha'a'}|\varrho_{\alpha'}(t)|f_{\alpha'a'}\rangle\,\mathrm{d}t = \pi \int \mathrm{d}E\, \chi_{E\Delta}^{(\alpha')}(E)(Ea'|\varrho_{\alpha'}|Ea'),$$

$$(21)\qquad \int_0^{\tau \ll (1/\Delta)} \mathrm{d}t \langle f_{\alpha'a'}|\varrho_{\alpha'}(t)|f_{\alpha'a'}\rangle \approx \tau \langle f_{\alpha'a'}|\varrho_{\alpha'}|f_{\alpha'a'}\rangle\,.$$

We will discuss the physical meaning of measurements represented by (20), (21) in the simple case when

$$\chi_{E\Delta}^{(\alpha')}(E') - \chi_{E\Delta}^{(\alpha)}(E') = \chi_{E\Delta}(E')\,.$$

Let us define

$$(22)\qquad \delta(E-E')(\alpha'a'\|S(E)\|\alpha a) = \langle \alpha'E'a'+|\alpha Ea-\rangle\,.$$

We then have from (18) and (20)

$$(23)\qquad \int_0^{\tau \gg (1/\Delta)} \mathrm{d}t \langle f_{\alpha'a'}|\varrho_{\alpha'}(t)|f_{\alpha'a'}\rangle \approx \int_{E-(1/2)}^{E+(1/2)} \mathrm{d}E' |(\alpha'a'\|S(E')\|\alpha a)|^2\,.$$

The integrand in (19) can be tought of as describing the signals from some detectors as they develop in the course of time; summing up all signals in a very long time interval as in (23) and collecting them together amounts to a measurement of integrated intensity in channel α'.

In other words this is a measurement of the transition probability from the initial state $|f_{\alpha a}(E\Delta)\rangle$ to the subspace of channel α' spanned by states $\chi_{E\Delta}^{(\alpha')}(E')|\alpha'E'a'\rangle$ with some fixed a'. The alternative possibility as given by (21)

from (18) and (22) reads

$$(24)\qquad \int_0^{\tau \ll (1/\Delta)} \mathrm{d}t \langle f_{\alpha' a'} | \varrho_{\alpha'}(t) | f_{\alpha' a'} \rangle \approx \tau \left| \int_{E-(\Delta/2)}^{E+(\Delta/2)} \mathrm{d}E' (\alpha' a' \| S(E') \| \alpha a) \right|^2 ,$$

i.e., apart from the factor τ in front, it gives the probability of finding the system described by $\varrho_{\alpha'}$ in the state $|f_{\alpha' a'}\rangle$ practically as soon as the outgoing waves begin to travel in channel α'.

It is quite clear that (24) just represents the contributions of undelayed processes (« instantaneous », according to De Dominicis' picture ([13])).

Also it turns out that it gives the transition probability from some element of the subspace spanned by the $|f_{\alpha a}(E\Delta)\rangle$ states to some elements of the subspace spanned by the $|f_{\alpha' a'}(E\Delta)\rangle$ states. Incidentally we remark, that, if the a, a' quantum numbers have discrete spectra, the subspaces spanned by the $|f_{\alpha' a'}(E\Delta)\rangle$, $|f_{\alpha a}(E\Delta)\rangle$, as E, Δ, a', a are varying, are Hilbert spaces.

In order to discuss more in detail our choice of the « allowed » channel states, let us now return to eq. (16) and insert it in eq. (24); we will consider the case when contributions rapidly varying with the energy are represented by the matrix element of $\mathscr{V}$ given in (17). Moreover we choose those intervals $[E; \Delta]$ that are contained in $[E_1; \Delta_1]$ and are such that the first term on the r.h.s. of (16), the factors multiplying $(E - E_\nu)^{-1}$ and the E_ν eigenvalues themselves all can be regarded as constants over that range.

We are interested in the case when, for all ν, $\mathrm{Im}\, E_\nu \ll \Delta$ and many levels are lying in the chosen energy range, *i.e.* the average level spacing $D = \Delta/N$ is much smaller than Δ, N being the number of levels in the range. Under these conditions we have

$$(25)\qquad \frac{1}{\Delta}\int \chi_{E\Delta}(E')\,\mathrm{d}E' (\alpha' a' \| S(E') \| \alpha a) = (\alpha' a' \| S_{\mathrm{av}}(E\Delta) \| \alpha a) =$$

$$= (a' \| a)\,\delta_{\alpha\alpha'} - 2\pi i \left[\langle \psi^{(+)}_{\alpha' E a'} | \mathscr{U}_\alpha | o \alpha E a \rangle - \frac{i\pi}{\Delta} \sum_n \langle \psi^{(-)}_{\alpha' E a'} | H_{oc} | cn \rangle \langle cn | H_{co} | \psi^{(-)}_{\alpha E a} \rangle \right].$$

As explained before the average transition amplitude has a quite well defined physical meaning, describing transitions in the subspace of the $|f_{\alpha a}(E\Delta)\rangle$ wave packets.

Now we look for states $|\Psi^{(+)}_{\alpha' E a'}\rangle$ belonging to the eigenvalue 1 of $P_{E\Delta}$ and such that

$$(26)\qquad (\alpha' a' \| S_{\mathrm{av}}(E\Delta) \| \alpha a) = (a' \| a)\,\delta_{\alpha\alpha'} - 2\pi i \langle \Psi^{(+)}_{\alpha' E a'} | [\mathscr{U}_\alpha + \overline{\mathscr{V}}_{E\Delta}] | o \alpha E a \rangle ,$$

with a suitable new interaction operator $\overline{\mathscr{V}}_{E\Delta}$.

([21]) The papers quoted under ([13]); H. Feshbach, C. Porter and V. Weisskopf: *Phys. Rev.*, **96**, 448 (1954).

If we compare with (7′) and (16) we easily realize that a sufficient condition for (26) to hold true is

$$|\Psi^{(+)}_{\alpha' E a'}\rangle = |\psi^{(+)}_{\alpha' E a'}\rangle + \frac{1}{E^{(-)} - K}\,\overline{\mathscr{V}}_{E\Delta}\,|\Psi^{(+)}_{\alpha' E a'}\rangle\,, \tag{27}$$

provided

$$\langle\Psi^{(+)}_{\alpha' E a'}|\,\overline{\mathscr{V}}_{E\,\Delta}\,|\psi^{(-)}_{\alpha E a}\rangle = -\frac{i\pi}{\Delta}\sum_n \langle\psi^{(+)}_{\alpha' E a'}|H_{oc}|cn\rangle\langle cn|H_{co}|\psi^{(-)}_{\alpha E a}\rangle\,.$$

From this it follows that a possible definition of $\overline{\mathscr{V}}_{E\Delta}$ is

$$\langle\Psi^{(+)}_{\alpha' E a'}|\,\overline{\mathscr{V}}_{E\,\Delta} = -\frac{i\pi}{\Delta}\sum_{c} \langle\psi^{(+)}_{\alpha' E a'}|H_{oc}|cn\rangle\langle cn|H_{co}\,. \tag{28}$$

From (28) and (27) it follows that $\langle\Psi^{(+)}_{\alpha' E a'}|$ satisfies the equation

$$\langle\Psi^{(+)}_{\alpha' E a'}|\,(E-K) = \langle\Psi^{(+)}_{\alpha' E a'}|\,\overline{\mathscr{V}}_{E\,\Delta} = \langle\Psi^{(+)}_{\alpha' E a'}|\sum_n H_{oc}|cn\rangle\, a_n\langle cn|H_{co}\,.$$

Following a procedure quite analogous to that of Feshbach it is easy to find for $\overline{\mathscr{V}}_{E\,\Delta}$ the expression

$$\overline{\mathscr{V}}_{E\,\Delta} = -\frac{i\pi}{\Delta}\sum_n H_{oc}|cn\rangle\left[1 - \frac{i\pi}{\Delta}\sum_{n'}\frac{\langle\psi^{(+)}_{\alpha' E a'}|H_{oc}|cn'\rangle}{\langle\psi^{(+)}_{\alpha' E a'}|H_{oc}|cn\rangle}\cdot\right.$$
$$\left.\cdot\,\langle cn'|H_{co}\,\frac{1}{E^{(+)} - K}\,H_{oc}|cn\rangle\right]^{-1}\langle cn|H_{co}\,. \tag{29}$$

Clearly we have here, in general, a dependence of $\overline{\mathscr{V}}_{E\Delta}$ on $\alpha' a'$, which will disappear when the dependence of the matrix element $\langle\psi^{(+)}_{\alpha' E a'}|H_{oc}|cn\rangle$ on $\alpha' a'$ will be essentially the same for all the intermediate decaying states $|cn\rangle$ in the $[E;\Delta]$ interval.

We have thus stated a condition for the usefulness of the definition (29) of $\overline{\mathscr{V}}_{E\,\Delta}$.

Clearly $\overline{\mathscr{V}}_{E\,\Delta}$ is not hermitian, thus expressing the fact that some fraction of the incoming flux is lost if we only take into account transitions in the subspace of the $|f_{\alpha a}(E\Delta)\rangle$ wave packets, corresponding to the instantaneous interactions of De Dominicis.

According to the previous discussion the losses are due to the long lived intermediate compound states, *i.e.* such that their lifetime is much greater than $\hbar/\Delta$.

We remember that the $|cn\rangle$ eigenstates of H_c appearing both in $\mathscr{H}_\alpha$ and in $\overline{\mathscr{V}}_{E\,\Delta}$, though lying in the subspace of the decaying states, are not to be identified with them (see comments after eq. (17)).

The contributions of virtual transitions to the subspace of the closed channels and back to the subspace of the open channels are present in the r.h.s. of (26) but only in so far as they are allowed, after exclusion of compound nucleus processes, just as one would expect according *e.g.* to Weisskopf's picture [1].

The last step, towards our definition of an « equivalent » two channel problem, involves the elimination of all open channels but those explicitly appearing as initial and final in (26). Let us define a projector P such that for both chosen channels

$$P\,|o\alpha Ea\rangle = |o\alpha Ea\rangle\,.$$

Introducing the notation:

$$(1-P)\,|\psi\rangle = |(0)\psi\rangle\,, \qquad P\,|\psi\rangle = |(1)\psi\rangle\,, \qquad \text{all } |\psi\rangle$$

$$P\mathcal{O}P = \mathcal{O}^{(1)}\,, \qquad (1-P)\,\mathcal{O}(1-P) = \mathcal{O}^{(0)}\,,$$

$$(1-P)\,\mathcal{O}P = \mathcal{O}^{(01)}\,, \qquad P\,\mathcal{O}(1-P) = \mathcal{O}^{(10)}\,, \qquad \text{all operators } \mathcal{O}$$

we have that the same procedure used for transforming eq. (5) into eq. (7) readily gives (we point out that P commutes with $P_{E_1 A_1}$ and with $P_{E\,1}$)

$$(30) \qquad \langle \Psi^{(+)}_{\alpha' E a'} | [\mathcal{U}_\alpha + \overline{\mathcal{V}}_{E\,1}] | o\alpha Ea\rangle =$$
$$= \langle (1)\Psi^{(+)}_{\alpha' E a'} | [\mathcal{U}^{(1)}_\alpha + \overline{\mathcal{V}}^{(1)}_{E\,1}] | (1) o\alpha Ea\rangle + \langle (0)\Psi^{(+)}_{\alpha' E a'} | [\mathcal{U}^{(01)}_\alpha + \overline{\mathcal{V}}^{(01)}_{E\,1}] | (1) o\alpha Ea\rangle\,.$$

We remark that (27) can also be written as

$$(27') \qquad \langle \Psi^{(+)}_{\alpha' E a'} | = \langle \psi^{(+)}_{\alpha' E a'} | \left[1 + \overline{\mathcal{V}}_{E\,1} \frac{1}{E^{(+)} - \mathcal{K}} \right],$$

with the definition

$$(31) \qquad \mathcal{K} = K + \overline{\mathcal{V}}_{E\,1} = H_{\alpha o} + \mathcal{U}_\alpha + \overline{\mathcal{V}}_{E\,1}\,.$$

Remembering the definition (15*a*) of $|\psi^{(+)}_{\alpha' E a'}\rangle$ we can write (27′) in the form:

$$\langle \Psi^{(+)}_{\alpha' E a'} | = \langle \alpha' E a' | \left[1 + \overline{\mathcal{V}}_{E\,1} \frac{1}{E^{(+)} - \mathcal{K}} \right] + \langle \alpha' E a' | \mathcal{U}_{\alpha'} \frac{1}{E^{(+)} - K} \left[1 + \overline{\mathcal{V}}_{E\,1} \frac{1}{E^{(+)} - \mathcal{K}} \right].$$

But, after (9) and (31),

$$\frac{1}{E^{(+)} - K} \left[1 + \overline{\mathcal{V}}_{E\,1} \frac{1}{E^{(+)} - \mathcal{K}} \right] = \frac{1}{E^{(+)} - \mathcal{K}}\,,$$

and thus we have

$$\langle \Psi^{(+)}_{\alpha' E a'} | = \langle \alpha' E a' | \left[1 + (\mathscr{U}_{\alpha'} + \overline{\mathscr{V}}_{E\Delta}) \frac{1}{E^{(+)} - \mathscr{K}} \right] . \tag{32}$$

Now the equations analogous to (11) and (12) read

$$\langle (0) \Psi^{(+)}_{\alpha' E a'} | = \langle (1) \Psi^{(+)}_{\alpha' E a'} | \mathscr{K}^{(10)} \frac{1}{E^{(+)} - \mathscr{K}^{(0)}} ,$$

$$\alpha' E a' | \mathscr{U}_{\alpha'} (1 - P) = \langle \alpha' E a' | K^{(10)} ,$$

and the analogous of (7) becomes, from (30),

$$\langle \Psi^{(+)}_{\alpha' E a'} | [\mathscr{U}_{\alpha} + \overline{\mathscr{V}}_{E\Delta}] | \alpha E a \rangle = \tag{33}$$

$$= \langle (1) \Psi^{(+)}_{\alpha' E a'} | \left[\mathscr{U}^{(1)}_{\alpha} + \overline{\mathscr{V}}^{(1)}_{E\Delta} + \mathscr{K}^{(10)} \frac{1}{E^{(+)} - \mathscr{K}^{(0)}} \mathscr{K}^{(01)} \right] | \alpha E a \rangle ,$$

with

$$\langle (1) \Psi^{(+)}_{\alpha' E a'} | = \langle \alpha' E a' | \left[1 + (\mathscr{U}^{(1)}_{\alpha} + \overline{\mathscr{V}}^{(1)}_{E\Delta}) \frac{1}{E^{(+)} - \mathscr{K}^{(1)}} \right] + \tag{32'}$$

$$+ \langle (1) \Psi^{(+)}_{\alpha' E a'} | \mathscr{K}^{(10)} \frac{1}{E^{(+)} - \mathscr{K}^{(0)}} \mathscr{K}^{(01)} \frac{1}{E^{(+)} - \mathscr{K}^{(1)}} .$$

The last two equations define the two channels equivalent problem we were looking for.

By the way $\langle (1) \Psi^{(+)}_{\alpha' E a'} |$ satisfies

$$\langle (1) \Psi^{(+)}_{\alpha' E a'} | \left[E - \mathscr{K}^{(1)} - \mathscr{K}^{(10)} \frac{1}{E^{(+)} - \mathscr{K}^{(0)}} \mathscr{K}^{(01)} \right] = 0 . \tag{34}$$

The last operator to the right in the bracket eq. (34) clearly describes transitions from the two chosen to the other open channels and back to the chosen ones; in addition to virtual transitions we have here also real transitions and thus the operator is in general non hermitian.

5. – As a preliminary remark to a discussion of the two channels equivalent problem defined at the very end of the previous section let us state that what we are interested in is a classification of different processes contributing to the average transition amplitude.

In fact this amplitude gives the effect of all undelayed or instantaneous transitions and we are faced with the question of giving a physical meaning to the various terms appearing in the r.h.s. of (33).

According to the philosophy we have explained at the beginning of this paper, the classification we are proposing has the essential feature of allowing a separation both formal and operational (scil. experimental) of the various contributions.

We point out that if no other channels but α, α' are open in the energy interval $[E; \Delta]$, then the third term in the r.h.s. of (33) is absent and we have to consider

$$(\alpha' a' \| S_{\mathrm{av}}(E\Delta) \| \alpha a) - (a' \| a)\, \delta_{\alpha\alpha'} = 2\pi i \langle \Psi^{(+)}_{\alpha' E a'} | [\mathscr{U}_\alpha + \overline{\mathscr{V}}_{E\Delta}] | \alpha E a \rangle \ .$$

Remembering then eq. (25) we have that the $\mathscr{U}_\alpha$ contribution can be isolated out being the second term on the r.h.s. of (25) and what remains is due to $\overline{\mathscr{V}}_{E\Delta}$.

The last contribution will be absent when we will not have closely spaced narrow resonances in the considered energy range giving rise to a term in the original transition amplitude being a wildly fluctuating function of the energy. We expect that such a term must be present in general in the neighbourhood of a giant resonance of the type of those known in nucleon-nucleus scattering and in photonuclear reactions. In turn we expect that sufficiently far from the giant resonance or in between two successive giant resonances the main contribution to the transition amplitude will be due to $\mathscr{U}_\alpha$.

We just remember that $\mathscr{U}_\alpha$ includes a Born approximation term and the distant levels contributions being the effect of the operator $\mathscr{U}$ of eq. (14).

Next let us show that the three different contributions just mentioned can be indeed formally separated from eq. (33).

First we point out that (32′) can be written as

$$\langle (1)\Psi^{(\pm)}_{\alpha' E a'} | = \langle \Phi^{(\pm)}_{\alpha' E a'} | + \langle (1)\Psi^{(\pm)}_{\alpha' E a'} | \mathscr{K}^{(10)} \frac{1}{E^{(\pm)} - \mathscr{K}^{(0)}} \mathscr{K}^{(01)} \frac{1}{E^{(\pm)} - \mathscr{K}^{(1)}} \,,$$

with

$$\langle \Phi^{(\pm)}_{\alpha' E a'} | = \langle \alpha' E a' | \left[1 + (\mathscr{U}^{(1)}_{\alpha'} + \overline{\mathscr{V}}^{(1)}_{E\Delta}) \frac{1}{E^{(\pm)} - \mathscr{K}^{(1)}} \right] \tag{35}$$

being the analogous of (13′) and (15a). Clearly the $\langle \Phi^{(\pm)}_{\alpha' E a} |$ only depend on $\mathscr{U}^{(1)}_{\alpha'}$ and $\overline{\mathscr{V}}^{(1)}_{E\Delta}$: Thus we can proceed in much the same way as for (16) and obtain

$$\langle (1)\Psi^{(+)}_{\alpha' E a'} | \left[\mathscr{U}^{(1)}_{\alpha} + \overline{\mathscr{V}}^{(1)}_{E\Delta} + \mathscr{K}^{(10)} \frac{1}{E^{(+)} - \mathscr{K}^{(0)}} \mathscr{K}^{(01)} \right] | \alpha E a \rangle = \tag{36}$$

$$= \langle (1)\Psi^{(+)}_{\alpha' E a'} | \mathscr{K}^{10} \frac{1}{E^{(+)} - \mathscr{K}^{(0)}} \mathscr{K}^{(01)} | \Phi^{(-)}_{\alpha E a} \rangle + \langle \Phi^{(+)}_{\alpha' E a'} | [\mathscr{U}^{(1)}_{\alpha} + \overline{\mathscr{V}}^{(1)}_{E\Delta}] | \alpha E a \rangle \ .$$

Next let us decompose the r.h.s. of (35) following backwards the way leading from (27) to (32); remembering (31) we have

$$\langle \Phi^{(\pm)}_{\alpha Ea}| = \langle \alpha Ea| \left[1 + \overline{\mathscr{V}}^{(1)}_{EA} \frac{1}{E^{(\pm)} - \mathscr{K}^{(1)}}\right] + \langle \alpha Ea| \mathscr{U}^{(1)}_{\alpha} \frac{1}{E^{(\pm)} - K^{(1)}} \left[1 + \overline{\mathscr{V}}^{(1)}_{EA} \frac{1}{E^{(\pm)} - \mathscr{K}^{(1)}}\right] =$$

$$= \langle \varphi^{(\pm)}_{\alpha Ea}| \left[1 + \overline{\mathscr{V}}^{(1)}_{EA} \frac{1}{E^{(\pm)} - \mathscr{K}^{(1)}}\right] = \langle \varphi^{(\pm)}_{\alpha Ea}| + \langle \Phi^{(\pm)}_{\alpha Ea}| \overline{\mathscr{V}}^{(1)}_{EA} \frac{1}{E^{(\pm)} - K^{(1)}},$$

with

$$\langle \varphi^{(\pm)}_{\alpha Ea}| = \langle \alpha Ea| \left[1 + \mathscr{U}^{(1)}_{\alpha} \frac{1}{E^{(\pm)} - K^{(1)}}\right].$$

It is then straightforward to obtain

$$\langle \Phi^{(+)}_{\alpha' Ea'}| [\mathscr{U}^{(1)}_{\alpha} + \overline{\mathscr{V}}^{(1)}_{EA}] |\alpha Ea\rangle = \langle \varphi^{(+)}_{\alpha' Ea'}| \mathscr{U}^{(1)}_{\alpha} |\alpha Ea\rangle + \langle \Phi^{(+)}_{\alpha' Ea'}| \overline{\mathscr{V}}^{(1)}_{EA} |\varphi^{(-)}_{\alpha Ea}\rangle ;$$

which, substituted into (36) gives finally

$$(37) \qquad \langle (1)\Psi^{(+)}_{\alpha' Ea'}| \left[\mathscr{U}^{(1)}_{\alpha} + \overline{\mathscr{V}}^{(1)}_{EA} + \mathscr{K}^{(10)} \frac{1}{E^{(+)} - \mathscr{K}^{(0)}} \mathscr{K}^{(01)}\right] |\alpha Ea\rangle =$$

$$\langle \varphi^{(+)}_{\alpha' Ea'}| \mathscr{U}^{(1)}_{\alpha} |\alpha Ea\rangle + \langle \Phi^{(+)}_{\alpha' Ea'}| \overline{\mathscr{V}}^{(1)}_{EA} |\varphi^{(-)}_{\alpha Ea}\rangle + \langle (1)\Psi^{(+)}_{\alpha' Ea'}| \mathscr{K}^{(10)} \frac{1}{E^{(+)} - \mathscr{K}^{(0)}} \mathscr{K}^{(01)} |\Phi^{(-)}_{\alpha Ea}\rangle .$$

According to what previously stated the third term on the r.h.s. of (37) is absent when the channels α, α' only are open, in the energy range $[E; A]$; the second one only is important in the neighbourhood of a giant resonance.

When no one of the three terms is absent or negligible, a different separation than (37) may prove useful, both for comparison with other theories and for applications: *i.e.* the one corresponding to separating out the effects dependent of $\overline{\mathscr{V}}_{EA}$, which according to what previously stated, are expecially important in the neighbourhood of a giant resonance. This separation can be done quite easily by just pointing out that, owing to (31) we are allowed to write

$$\mathscr{K}^{(10)} \frac{1}{E^{(+)} - \mathscr{K}^{(0)}} \mathscr{K}^{(01)} = K^{(10)} \frac{1}{E^{(+)} - K^{(0)}} K^{(01)} + O(\overline{\mathscr{V}}_{EA}) ,$$

with the second term on the r.h.s.; collecting the whole of the $\overline{\mathscr{V}}_{EA}$ dependence and thus being negligible but in the neighbourhood of a giant resonance.

Let us now define

$$\mathscr{U}^{(1)}_{\alpha} + K^{(10)} \frac{1}{E^{(\pm)} - K^{(0)}} K^{(01)} = \mathscr{H}^{(\pm)}_{\alpha} ,$$

$$\mathscr{U}^{(1)}_{\alpha} + \overline{\mathscr{V}}^{(1)}_{EA} + \mathscr{K}^{(10)} \frac{1}{E^{(\pm)} - \mathscr{K}^{(0)}} \mathscr{K}^{(01)} = \mathscr{H}^{(\pm)}_{\alpha} + \overline{\mathscr{H}}^{(\pm)} .$$

From (15*a*) and (27′) we have

$$\langle(1)\Psi^{(\pm)}_{\alpha Ea}|_{\overline{\mathscr{V}}_{EA}=0} = \langle(1)\psi^{(\pm)}_{\alpha Ea}| .$$

With some lengthy but straightforward algebra it is not difficult to obtain then

$$\langle(1)\psi^{(\pm)}_{\alpha Ea}| = \langle \alpha Ea| \left[1 + \mathscr{H}^{(\pm)}_{\alpha} \frac{1}{E^{(\pm)} - K^{(1)} - \mathscr{H}^{(\pm)}_{\alpha} + \mathscr{U}^{(1)}_{\alpha}}\right],$$

and

$$\langle(1)\Psi^{(\pm)}_{\alpha Ea}| = \langle(1)\psi^{(\pm)}_{\lambda Ea}| + \langle(1)\Psi^{(\pm)}_{\alpha Ea}|\overline{\mathscr{H}^{(\pm)}} \frac{1}{\overline{E^{(\pm)} - K^{(1)} - \mathscr{H}^{(\pm)}_{\lambda} + \mathscr{U}^{(1)}_{\lambda}}}.$$

In the same way as for (16) one then has

$$\begin{aligned}(38)\qquad \langle(1)\Psi^{(+)}_{\alpha' Ea'}|[\mathscr{H}^{(+)}_{\alpha} + \overline{\mathscr{H}^{(\cdot)}}]|\alpha Ea\rangle &= \langle(1)\Psi^{(+)}_{\lambda' Ea'}|\mathscr{H}^{(+)}_{\alpha}|\alpha Ea\rangle + \\ &+ \langle(1)\Psi^{(+)}_{\alpha' Ea'}|\overline{\mathscr{H}^{(+)}}\left|\,|(1)\psi^{(-)}_{\alpha Ea}\rangle - \frac{1}{E^{(-)} - K^{(1)} - \mathscr{H}^{(+)}_{\lambda} + \mathscr{U}^{(1)}_{\lambda}}\mathscr{H}^{(+)}_{\alpha}|\alpha Ea\rangle\right| \cdot \\ &= \langle(1)\psi^{(+)}_{\lambda' Ea'}|\mathscr{H}^{(+)}_{\lambda}|\alpha Ea\rangle + \langle(1)\Psi^{(+)}_{\alpha' Ea'}|\overline{\mathscr{H}^{(+)}}|(1)\psi^{(-)}_{\alpha Ea}\rangle ,\end{aligned}$$

where manifestly the second term will be negligible with $\overline{\mathscr{V}}_{EA}$.

6. – The formal theory given in the previous sections can be used for practical calculations in a variety of different ways: let us mention here that if one regards the channel states as known, they can be used to construct a spectral representation of the channel hamiltonians H_α.

Then one can either postulate some form of the total hamiltonian H and look at its consequences, or start from the very beginning to work in the chosen subspace and try to determine relations amongst different observable quantities, as implied from the theory without any particular choice of H.

We point out that such relations exist and in fact are of non trivial nature: they are mathematically connected in part with the role of the intermediate states αn in describing the structure of the direct interactions, in part with general properties of the average transition amplitude as a function of the energy in the complex energy plane ([22]).

([22]) E. P. Wigner: *Proc. Camb. Phil. Soc.*, **47**, 790 (1951); *Ann. Math.*, **53**, 36 (1951); **55**, 7 (1952); E. P. Wigner and J. v. Neumann: *Ann. Math.*, **59**, 418 (1954).

On the physical side the existence of these relations has to do essentially with the fact that any unstable state can be experimentally defined in terms of the processes including transitions to and from it as intermediate virtual steps. And conversely the property of including transitions to and from some intermediate unstable state can be used not only to classify processes but also to state physically significant relations amongst them, at least in so far as one is able to select somehow [23] the contributions to any process due to the mentioned transitions.

We remark that owing to our use of a general representation, subject to the only condition of the total energy being diagonal, it is particularly simple to give a field theoretical analysis of the transition amplitudes.

As concluding remarks we will add here some general results that can be deduced from the theory without entering in the details of either the Hamiltonian or the relevant states.

Let us consider the case of s-wave elastic scattering of nucleons from nuclei neglecting spin: in this case the αE labels only are needed to classify the states.

For simplicity we will assume that no other channels are open. In this case formula (38) becomes identical with (30) and we are allowed to write for the original scattering amplitude, from (7′) and (16),

$$\langle \alpha E' + | \alpha E - \rangle = \delta(E' - E)[1 - 2\pi i(\langle \psi^{(+)}_{\alpha E} | \mathscr{U}_\alpha | \alpha E \rangle + \langle o\alpha E + | \mathscr{V} | \psi^{(+)}_{\alpha E} \rangle)] =$$

$$= \delta(E' - E) \cdot S \,.$$

We point out that the normalization of the $|\psi^{(+)}_{\alpha E}\rangle$ vectors can be chosen so that

$$\langle \psi^{(+)}_{\alpha E'} | \psi^{(-)}_{\alpha E} \rangle = \delta(E' - E) \left[1 - 2\pi i \langle \psi^{(+)}_{\alpha E} | \mathscr{U}_\alpha | \alpha E \rangle\right]. \tag{39}$$

The scalar products (39) can be regarded as elements of a unitary matrix which describes the scattering due to the interaction operator $\mathscr{U}_\alpha$. We will write

$$\langle \psi^{(+)}_{\alpha E'} | \psi^{(-)}_{\alpha E} \rangle = \delta(E' - E) \exp[2i\delta],$$

with

$$\delta = \delta(\alpha E) \qquad \text{and} \qquad \operatorname{Im} \delta = 0 \,.$$

[23] See *e.g.* G. F. Chew and F. E. Low: *Phys. Rev.*, **113**, 1640 (1959); M. Cini and S. Fubini: *Ann. Phys.* **10**, 352, (1960).

Next we can write for the average transition amplitude $\bar{S}$, from (25),

$$(40)\qquad \bar{S} = 1 - 2\pi i \left[\langle \psi^{(+)}_{\alpha E} | \mathscr{U}_\alpha | \alpha E \rangle - \frac{i\pi}{\Delta} \sum_n \langle \psi^{(+)}_{\alpha E} | H_{oc} | cn \rangle \langle cn | H_{co} | \psi^{(-)}_{\alpha E} \rangle \right] \cdot$$

$$\cdot \exp[2i\delta] \left[1 - \frac{2\pi^2}{\Delta} \sum_n \left| \langle \psi^{(+)}_{\alpha E} | H_{oc} | cn \rangle \right|^2 \right] = \exp[2i\delta] \left[1 - 2\pi i \langle \psi^{(+)}_{\alpha E} | \overline{\mathscr{V}}_{E\Delta} | \psi^{(+)}_{\alpha E} \rangle \right],$$

where use has been made of

$$|\psi^{(-)}_{\alpha E}\rangle = \int \mathrm{d}E' |\psi^{(+)}_{\alpha E'}\rangle \langle \psi^{(+)}_{\alpha E'} | \psi^{(-)}_{\alpha E}\rangle = |\psi^{(+)}_{\alpha E}\rangle \exp[2i\delta] .$$

If we define

$$(41)\qquad \frac{2\pi^2}{\Delta} \sum_n \left| \langle \psi^{(+)}_{\alpha E} | H_{oc} | cn \rangle \right|^2 = \gamma^2 > 0 ,$$

the condition that $\bar{S}$ does not contain all contributions from processes connecting the initial to the final state implies

$$1 > |\bar{S}| = |1 - \gamma^2| ,$$

i.e.

$$(42)\qquad \gamma^2 < 2 .$$

Now after writing (29) as

$$\overline{\mathscr{V}}_{E\Delta} = \sum_n H_{oc} |cn\rangle a_n \langle cn | H_{co} ,$$

we have that in our case a_n is given by

$$a_n = -i \frac{\pi}{\Delta} \left[1 - \frac{i\pi}{\Delta} \sum_{n'} \left(-\pi i \left| \langle \psi^{(+)}_{\alpha E} | H_{oc} | cn' \rangle \right|^2 + \right. \right.$$

$$\left. \left. + \frac{\langle \psi^{(+)}_{\alpha E} | H_{oc} | cn' \rangle}{\langle \psi^{(+)}_{\alpha E} | H_{oc} | cn \rangle} \mathrm{P.V.} \int \mathrm{d}E' \frac{\langle cn' | H_{co} | \psi^{(-)}_{\alpha E'} \rangle \langle \psi^{(-)}_{\alpha E'} | H_{oc} | cn \rangle}{E - E'} \right) \right]^{-1} ,$$

where the third term to the right inside the square bracket can be shown to be real.

The condition (42) is equivalent to

$$\frac{1}{2}\gamma^2 - 1 = \frac{\pi}{\Delta} \frac{\mathrm{Im}\, a_n}{|a_n|^2} < 0 ,$$

i.e., essentially, for any state $|\psi\rangle$,

$$\operatorname{Im}\langle\psi|\overline{\mathscr{V}}_{E\Delta}|\psi\rangle < 0\,.$$

In order to find which effects the imaginary part of $\overline{\mathscr{V}}_{E\Delta}$ has on the scattering states $|\psi^{(\pm)}_{\alpha E}\rangle$, we point out that with the definition

$$|\overset{0}{\Psi}{}^{(\pm)}_{\alpha E}\rangle = |\Psi^{(\pm)}_{\alpha E}\rangle_{\operatorname{Im} a_n = 0}\,,$$

we have

$$\bar{S} = 1 - 2\pi i\,[\langle\overset{0}{\Psi}{}^{(+)}_{\alpha E}|(\mathscr{H}_\alpha + \operatorname{Re}\overline{\mathscr{V}}_{E\Delta})|\alpha E\rangle + i\langle\Psi^{(+)}_{\alpha E}|\operatorname{Im}\overline{\mathscr{V}}_{E\Delta}|\overset{0}{\Psi}{}^{(-)}_{\alpha E}\rangle] =$$

$$= \exp[2i\overset{0}{\delta}][1 + 2\pi\langle\Psi^{(+)}_{\alpha E}|\operatorname{Im}\overline{\mathscr{V}}_{E\Delta}|\overset{0}{\Psi}{}^{(+)}_{\alpha E}\rangle]\,,$$

being

$$\exp[2i\overset{0}{\delta}] = 1 - 2\pi i\langle\overset{0}{\Psi}{}^{(+)}_{\alpha E}|(\mathscr{H}_\alpha + \operatorname{Re}\overline{\mathscr{V}}_{E\Delta})|\alpha E\rangle$$

and

$$|\overset{0}{\Psi}{}^{(+)}_{\alpha E}\rangle \exp[2i\overset{0}{\delta}] = |\overset{0}{\Psi}{}^{(-)}_{\alpha E}\rangle\,.$$

Moreover it is

$$\exp[2i\overset{0}{\delta}] = \exp[2i\delta][1 - 2\pi i\langle\overset{0}{\Psi}{}^{(+)}_{\alpha E}|\operatorname{Re}\overline{\mathscr{V}}_{E\Delta}|\psi^{(+)}_{\alpha E}\rangle]\,. \tag{43}$$

We easily obtain from (40) and (43)

$$\frac{\langle\Psi^{(+)}_{\alpha E}|\overline{\mathscr{V}}_{E\Delta}|\psi^{(+)}_{\alpha E}\rangle}{\langle\overset{0}{\Psi}{}^{(+)}_{\alpha E}|\operatorname{Re}\overline{\mathscr{V}}_{E\Delta}|\psi^{(+)}_{\alpha E}\rangle} = \frac{\gamma^2}{1 - \exp[2i(\overset{0}{\delta} - \delta)]}\,, \tag{44}$$

having assumed $\sin(\overset{0}{\delta} - \delta) \neq 0$. Now we point out that (44) allows the order of magnitude estimate

$$\left|\frac{\langle\Psi^{(+)}_{\alpha E}|\overline{\mathscr{V}}_{E\Delta}|\psi^{(+)}_{\alpha E}\rangle}{\overset{0}{\Psi}{}^{(+)}_{\alpha E}|\operatorname{Re}\overline{\mathscr{V}}_{E\Delta}|\psi^{(+)}_{\alpha E}\rangle}\right| \approx \left|\frac{\langle\Psi^{(+)}_{\alpha E}|H_{oc}|c\bar{n}\rangle}{\langle\overset{0}{\Psi}{}^{(+)}_{\alpha E}|H_{oc}|c\bar{n}\rangle}\right| \left|\frac{a_{\bar{n}}}{\operatorname{Re} a_{\bar{n}}}\right|, \tag{45}$$

From (45) and (44) we readily obtain

$$\frac{\gamma^2}{2|\sin(\overset{0}{\delta} - \delta)|}\left|\frac{\operatorname{Re} a_{\bar{n}}}{a_{\bar{n}}}\right| \approx \left|\frac{\langle\Psi^{(+)}_{\alpha E}|H_{oc}|c\bar{n}\rangle}{\langle\overset{0}{\Psi}{}^{(+)}_{\alpha E}|H_{oc}|c\bar{n}\rangle}\right|. \tag{46}$$

Let us remark that $|\langle\psi|H_{oc}|c\bar{n}\rangle|^2$ is a measure of the coupling of the general state ψ to the subspace in which the unstable states are lying which originate narrow resonances in the $[E\,;\,\Delta]$ energy range according to eq. (17).

We have thus found that this coupling, as it follows from (46), will be less and less effective for $\Psi^{(-)}$ than for $\overset{0}{\Psi}{}^{(-)}$ as γ^2 becomes vanishingly small, far from a giant resonance. In fact, in this case, $\overset{0}{\delta} - \delta \propto \gamma^2$, $|\mathrm{Re}\, a_{\bar{n}}/a_{\bar{n}}| \propto \gamma^2$ and this amounts to have

$$\frac{\Psi^{(-)}_{\alpha E}|H_{oc}|c\bar{n}}{\overset{0}{\Psi}{}^{(+)}_{\alpha E}|H_{oc}|c\bar{n}} \propto \gamma^2 .$$

Similarly, in the neighbourhood of a giant resonance we expect that $\overset{0}{\delta}$ is varying through $-\pi/2$, while γ^2 is definitely different from zero, and so $|\delta| \ll |\overset{0}{\delta}|$.

As γ^2 diminishes we have that $|\sin(\overset{0}{\delta} - \delta)| \approx 1$ and $|\mathrm{Re}\, a_{\bar{n}}/a_{\bar{n}}|$ becomes smaller and the same is true for the ratio on the r.h.s. of (46). We point out that the just obtained results can be stated by saying that both in the neighbourhood of a giant resonance and far from it, that ratio becomes smaller as $|\mathrm{Im}\, \overline{\mathscr{V}}_{EA}|$ increases. This corresponds clearly to the well known effect of the imaginary part of the complex potential well of decreasing the amplitude of the scattering wave function inside the range of the well.

This range is essentially the «interaction region» being in fact the region of the configuration space in which the «compound states» are used to analyze the general scattering state in the conventional dispersion theories.

Also we remark that, from (41), if γ^2 goes to zero, this means that the coupling of the $\psi^{(+)}_{\lambda E}$ states to the $|cn\rangle$ ones becomes vanishingly small, or, otherwise stated, $\psi^{(+)}_{\alpha E}$ becomes orthogonal to the subspace in which the $H_{oc}|cn\rangle$ are lying: since in this limit $\bar{S} \to \exp[2i\delta]$ *i.e.* the scattering depends on $\langle\psi^{(+)}_{\lambda E}|\mathscr{H}_\alpha|\alpha E$ only, it turns out that this matrix element essentially depends on the components of $\mathscr{H}_\alpha|\alpha E\rangle$ orthogonal to that subspace. If expressed in terms of the conventional theory the last statement is equivalent to classifying the $\langle\psi^{(-)}_{\lambda E}|\mathscr{H}_\alpha|\alpha E\rangle$ matrix element far from the giant resonance as the surface contribution to the direct transition amplitude $\bar{S}$.

When other channels are open eq. (33) can be written as (38), putting in evidence a term vanishing with $\overline{\mathscr{V}}_{EA}$.

Clearly as the vanishing of $\overline{\mathscr{V}}_{EA}$ is connected to the vanishing of γ^2, in so far as γ^2 can be given by eq. (41) the first term on the r.h.s. of (38) also is a surface contribution, owing to its equivalence to $\langle\psi^{(+)}_{\alpha' Ea'}|\mathscr{H}_\alpha|\alpha Ea\rangle$, immediately following by comparison of (33) and (38) with $\overline{\mathscr{V}}_{EA} = 0$.

The domain of applicability of our result on the surface contribution to the direct transition amplitude is limited by the conditions stated before eq. (25), allowing us to obtain a simple form for the result of the energy averaging on the near levels contribution to the original amplitude.

We point out that in fact such conditions are certainly not necessary to obtain eq. (46) and the statements about the relative strength of the coupling

of $\overset{0}{\Psi}{}^{(+)}$ and $\Psi^{(+)}$ to the compound states: these statements do not depend on the special form (41) of γ^2. So we are allowed to state, from (46), that when $|\mathrm{Re}\, a_n^-/a_n^-| \to 0$ the vanishing of the $\overline{\mathscr{V}}_{E\lambda}$ contributions is essentially due to $\Psi^{(+)}$ becoming orthogonal to the space of the $H_{oc}|cn\rangle$ vectors and the resulting equality

$$\langle \Psi^{(+)}_{\alpha E} | \mathscr{U}_\alpha | \alpha E \rangle \approx \langle \psi^{(+)}_{\alpha E} | \mathscr{U}_\alpha | \alpha E$$

allows us to state that the residual contribution is due to the components of $\langle \Psi^{(+)}_{\alpha E}|$ orthogonal to the space of the $H_{oc}|cn\rangle$ states, *i.e.* to the « surface » overlap of $\langle \Psi^{(+)}_{\alpha E}|$ and $\mathscr{U}_\alpha|\alpha E\rangle$.

Clearly when the same set of $H_{oc}|cn\rangle$ states is involved in a transition amplitude connecting any two channels open in the energy interval $[E; \Delta]$, some similarities are expected to appear relating the transition amplitudes the one to the other. An aspect of such relations might be the existence of surface effects in different direct reactions.

Accordingly we expect that the present formalism is particularly suited for investigating the relations between the surface direct contributions appearing in pick-up reactions ([24]), in inelastic scattering of nucleons by nuclei, and in photonuclear reactions ([7,25]).

We stress that this is not the main type of relations which can be studied in the framework of the present theory: we point out that the theory lends itself quite naturally to a field theoretical analysis of the direct scattering amplitudes, without the difficulties connected with the conventional configuration space dispersion theories. Moreover, while including the main results about the distant levels contribution to the direct reactions so far obtained with the Kapur-Peierls dispersion theory ([4]), it turns out that it provides a framework well suited for discussing direct effects at giant resonance as well as far from it, and in fact so as to allow a simple interpretation both formally as well as from a physical point of view.

* * *

It is a pleasure to thank Prof. R. M. Williamson (Duke University - N.C.) and Dr. M. Nagasaky (Rykkyo University - Tokyo) for interesting discussions on this subject during their stay at C.S.F.N.

([24]) S. T. Butler quoted under ([14]); N. Austern, S. T. Butler and H. McManus: *Phys. Rev.*, **92**, 350 (1953).

([25]) A. M. Lane quoted under ([8]); A. M. Lane and J. E. Lynn: *Nucl. Phys.*, **11**, 646 (1959).

Some illuminating remarks and suggestions by Prof. C. Bloch (C.E.N., Saclay) and by Prof. M. Cini (Rome University) are gratefully acknowledged.

One of us (A.A.) is very indebted to Prof. C. Bloch for the kind hospitality at C.E.N., Saclay, where this work was begun.

RIASSUNTO

Viene presentata una teoria a dispersione delle reazioni nucleari che è una generalizzazione di quelle esistenti. La teoria è centrata attorno alla possibilità di definire una classe di problemi equivalenti all'originario problema di scattering nel senso che le loro soluzioni determinano tutte gli stessi elementi di matrice di transizione. Questi problemi equivalenti sono generati proiettando opportunamente quello originario su un sottospazio dello spazio lineare definito dagli autostati del continuo dell'hamiltoniana totale, che include gli stati di canale iniziali e finali. Una derivazione della formula di Breit-Wigner analoga a quella di Feshbach può essere data nella nostra teoria. I processi diretti vengono definiti mediando rispetto all'energia gli elementi di matrice di transizione, in modo che risultino eliminati i contributi dovuti agli stati « composti » di vita media lunga. Nel caso dello scattering nucleone-nucleo si ottiene in tal modo una definizione del modello ottico che, sebbene del tutto analoga a quelle convenzionali, però sembra più facilmente analizzabile con metodi di teoria dei campi. Viene altresì mostrato che i ben noti contributi di superficie alle reazioni dirette possono essere isolati in modo del tutto « naturale » dagli altri contributi diretti.

Poles in Production Cross Sections.

R. Ascoli

Istituto di Fisica dell'Università - Torino
Istituto Nazionale di Fisica Nucleare - Sezione di Torino

(ricevuto il 27 Luglio 1960)

Summary. — In this work we have investigated the nature and the position of the one-particle singularities that are expected in the production cross section $\partial^2\sigma/\partial w^2\,\partial\Delta^2$, introduced by Chew and Low, when considered as function of the momentum transfer Δ^2. We find that these singularities are in general in the complex Δ^2 plane. Furthermore we find a phenomenon analogous to the « anomalous tresholds » of the scattering processes. The phenomenon occurs only with strange particles or nuclei. In some cases it may prevent the application of the extrapolation procedure that has been proposed by Chew and Low to get scattering cross sections from production experiments.

1. – Introduction.

This kinematical investigation deals with the kind and position of the one-particle singularities that are expected in the production cross section $\partial^2\sigma/\partial w^2\partial\Delta^2$, introduced by Chew and Low [1], considered as function of the momentum transfer.

The results are particularly discussed in view of the extrapolation procedure proposed by Chew and Low [1] to get scattering cross sections from suitable production experiments.

Of course some more generally interesting phoenomena appear.

[1] G. F. Chew and F. E. Low: *Phys. Rev.*, **115**, 1640 (1959).

2. – Singularities of the production cross section $\partial^2\sigma/\partial w^2\partial\Delta^2$ as function of Δ^2.

We consider a process where a particle of mass m and momentum p collides with a particle of mass μ and momentum k producing a particle of mass m' and momentum p' and two further particles of masses μ' and μ'' and momenta k', k'' respectively.

Then we introduce as dynamical variables the square of the total energy $W^2 = (p + k)^2$, the momentum transfer between two particles $\Delta^2 = -(p - p')^2/4$, the square of the total mass $w^2 = (k' + k'')^2$ of the system of the particles k' and k''.

These variables are enough to define the cross section $\partial^2\sigma/\partial w^2\partial\Delta^2$, introduced firstly by CHEW and LOW [1].

Our programm will be to consider this cross section as function of Δ^2 at constant W^2 and w^2, to investigate what singularities are expected from the contribution of one-particle states. To this purpose we must firstly put into explicit form the connection between the cross section and the amplitude; we have

$$\sigma = \frac{\pi^2}{\sqrt{(kp)^2 - m^2\mu^2}} \int |T|^2 \delta^{(4)}(k' + k'' + p' - k - p)\, \delta(k'^2 - \mu'^2)\, \delta(k''^2 - \mu''^2) \cdot \tag{1}$$
$$\cdot \delta(p'^2 - m'^2)\, \theta(k')\, \theta(k'')\, \theta(p')\, \mathrm{d}^4k'\, \mathrm{d}^4k''\, \mathrm{d}^4p' .$$

After integration over k'_0, p'_0 and k'' we introduce polar angles θ', φ' for the vector $\boldsymbol{k}'$ and θ_0, φ_0 for the vector $\boldsymbol{p}'$. Then we obtain, after integration over $|\boldsymbol{k}'|$,

$$\sigma = \frac{\pi^2}{8\sqrt{(kp)^2 - m^2\mu^2}} \int |T|^2 \frac{|\boldsymbol{k}'|^2 |\boldsymbol{p}'|^2}{k'_0 k''_0 p'_0}\, \mathrm{d}\cos\theta'\, \mathrm{d}\varphi'\, \mathrm{d}|\boldsymbol{p}'|\, \mathrm{d}\cos\theta_0\, \mathrm{d}\varphi_0\, \frac{\mathrm{d}|\boldsymbol{k}'|}{\mathrm{d}W'} , \tag{2}$$

where

$$W' = \sqrt{|\boldsymbol{k}|^2 + \mu'^2} + \sqrt{|\boldsymbol{k} + \boldsymbol{p} - \boldsymbol{k}' - \boldsymbol{p}'|^2 + \mu''} + \sqrt{|\boldsymbol{p}'|^2 + m'^2} , \tag{3}$$

and all quantities are calculated on the energy shell.

Now it is particularly convenient to do the calculations in the reference system $\boldsymbol{k}' + \boldsymbol{k}'' = 0$. Furthermore we take the vector $\boldsymbol{k}$ as polar axis.

Owing to rotation invariance all quantities depend from φ' and φ_0 only through their difference $\varphi = \varphi' - \varphi_0$. So we take as parameters W, w, Δ^2, θ', φ.

Then we trasform the integrations over $|\boldsymbol{p}'|$ and $\cos\theta_0$ in integrations over w^2 and Δ^2. This is possible because $|\boldsymbol{p}'|$ and $\cos\theta_0$ depend only on Δ^2, w^2 and W^2. Actually $|\boldsymbol{p}'|$ depends only on w^2 and W^2. In fact we have

$$p'_0 = \frac{p'(k' + k'')}{w} = \frac{(p'_0(k'_0 + k''_0) + |\boldsymbol{p}'|^2)_{\mathrm{c.m.}}}{w} , \tag{4}$$

where the quantities in the last expression refer to the center of mass system. Now in the center of mass system p_0', $k_0' + k_0''$ and $|\boldsymbol{p}'|$ are functions of W^2 and w^2 only. So p_0' and $|\boldsymbol{p}'|$ (in the system $\boldsymbol{k}' + \boldsymbol{k}'' = 0$) are functions of W and w^2 only. For $\cos\theta_0$ we have

$$\cos\theta_0 = \frac{p_0' k_0 - p'k}{|\boldsymbol{p}'||\boldsymbol{k}|}. \tag{5}$$

By squaring the relation $k' + k'' = p - p' + k$ we obtain immediately $p'k$ as linear function of W^2, w^2, Δ^2. We have already seen that p_0' and $|\boldsymbol{p}'|$ are functions of W^2 and w^2 only. Now from $|\boldsymbol{k}| = |\boldsymbol{p} - \boldsymbol{p}'|$, $k_0 + (p_0 - p_0') = w$ and $(p - p')^2 = -4\Delta^2$ we have

$$k_0 = \frac{w^2 + \mu^2 + 4\Delta^2}{2w}, \qquad p_0 - p_0' = \frac{w^2 - 4\Delta^2 - \mu^2}{2w}, \tag{6}$$

so that also k_0 and $|\boldsymbol{k}|$ depend only on w^2 and Δ^2. So we conclude from formula (5) that $\cos\theta_0$ is a function of W^2, w^2 and Δ^2 only: actually $\cos\theta_0$ as function of Δ^2 has the form $f(\Delta^2)/\sqrt{k_0^2 - \mu^2}$, where $f(\Delta^2)$ is a polynomial in Δ^2 and k_0 is given by (6): so $\cos\theta_0$ is a regular function of $(p - p')^2$ except a cut between the points $(w - \mu)^2$ and $(w + \mu)^2$.

So (2) becomes

$$\sigma = \frac{\pi^3}{4\sqrt{(kp)^2 - m^2\mu^2}} \int |T|^2 \frac{|\boldsymbol{k}'|^2 |\boldsymbol{p}'|^2}{k_0' k_0'' p_0'} \frac{\partial |\boldsymbol{p}'|}{\partial w^2} \frac{\partial \cos\theta_0}{\partial \Delta^2} \frac{\mathrm{d}|\boldsymbol{k}'|}{\mathrm{d}W'} \,\mathrm{d}\cos\theta' \,\mathrm{d}\varphi \,\mathrm{d}w^2 \mathrm{d}\Delta^2. \tag{7}$$

Now we have from $|\boldsymbol{k}'| = |\boldsymbol{k}''|$ and $k_0' + k_0'' = w$

$$k_0' = \frac{w^2 + \mu'^2 - \mu''^2}{2w}, \qquad k_0'' = \frac{w^2 + \mu''^2 - \mu^2}{2w}, \tag{8}$$

so that k_0' and k_0'' depend only on w^2. We have already seen that p_0', $|\boldsymbol{p}'|$ and $\partial|\boldsymbol{p}'|/\partial w^2$ depend only on W^2 and w^2 and one easily reaches the same conclusion for $\mathrm{d}|\boldsymbol{k}|/\mathrm{d}W'$, using (3) and (8). Then all the quantities appearing in the integrand, apart from T, depend only on W^2, w^2, Δ^2; Δ^2 enters only in $\partial\cos\theta_0/\partial\Delta^2$ in the above given form. So it is finally

$$\frac{\partial^2\sigma}{\partial w^2 \partial \Delta^2} = \frac{\pi^3}{4\sqrt{(kp)^2 - m^2\mu^2}} \frac{|\boldsymbol{k}'|^2 |\boldsymbol{p}'|^2}{k_0' k_0'' p_0'} \frac{\partial |\boldsymbol{p}'|}{\partial w^2} \frac{\partial \cos\theta_0}{\partial \Delta^2} \frac{\mathrm{d}|\boldsymbol{k}'|}{\mathrm{d}W'} \int |T|^2 \mathrm{d}\cos\theta' \,\mathrm{d}\varphi = \tag{9}$$

$$= \frac{F(W^2, w^2, \Delta^2)}{\sqrt{k_0^2 - \mu^2}} \int |T|^2 \,\mathrm{d}\cos\theta' \,\mathrm{d}\varphi,$$

where $F(W^2, w^2, \Delta^2)$ is a regular function of Δ^2.

We are now interested in the singularities of $\partial^2\sigma/\partial w^2\partial\Delta^2$ as function of Δ^2, particularly in connection with the extrapolation procedure proposed by CHEW and LOW (1). So we limit ourselves to the contributions that become infinite as $(p-p')^2=\mu_0^2$, where μ_0 is the lowest mass of the particles leading to a singularity of this type. In the perturbative calculation these contributions correspond to the Feynman graphs of Fig. 1; more generally they

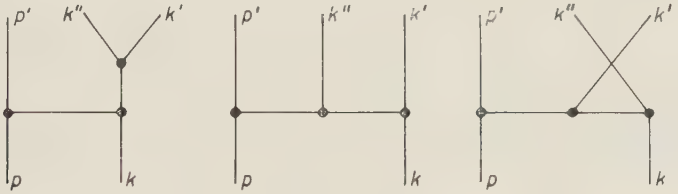


Fig. 1. – The lowest order Feynman graphs that, in perturbation theory, lead to the contributions that become infinite for $(p-p')^2=\mu_0^2$ (μ_0 is the mass of the intermediate particle concurring in one vertex with the particles of momenta p and p'). Time flows upwards.

correspond to poles occurring in the amplitude for $(k'+k'')^2=\mu_i^2$, $(k-k')^2=\mu_j^2$ $(k-k'')^2=\mu_k^2$, where μ_i, μ_j, μ_k are the possible masses in each case. The first poles do not concern the variable Δ^2; the remaining two kinds of poles have the same form, so we limit ourselves to study the effect of a pole occurring for $(k-k')^2=\mu_1^2$.

We remark firstly that $\partial^2\sigma/\partial w^2\partial\Delta^2$ has in any case a cut between the points $\Delta^2=(w-\mu)^2$ and $\Delta^2=(w+\mu)^2$, due to the factor $1/\sqrt{k_0^2-\mu^2}$ in the formula (9). This cut may be easily eliminated in any case, for instance dividing $\partial^2\sigma/\partial w^2\partial\Delta^2$ by $\sqrt{k_0^2-\mu^2}$: actually the quantity considered by Chew and Low for the purpose of extrapolation contains such a factor. We do not take further into account this singularity.

The poles occurring in the amplitude at $(k-k')^2=\mu_1^2$ lead to pure and to crossed terms in the cross section: they contain integrals of the types

$$\text{(10)}\qquad \begin{cases} I_1=\int\dfrac{\mathrm{d}\cos\theta'\,\mathrm{d}\varphi}{((k-k')^2-\mu_1^2)^2}, \qquad I_2=\int\dfrac{T_1\,\mathrm{d}\cos\theta'\,\mathrm{d}\varphi}{(k-k')^2-\mu_1^2}, \\ I_3=\int\dfrac{\mathrm{d}\cos\theta'\,\mathrm{d}\varphi}{((k-k')^2-\mu_1^2)\,((k-k'')^2-\mu_2^2)}, \end{cases}$$

where T_1, is a contribution to T which does not contain poles in Δ^2.

Let us discuss the first integral: the denominator is of the type:

$$\text{(11)}\qquad (k-k')^2-\mu_1^2=M_1-2k_0k_0'+2|\boldsymbol{k}||\boldsymbol{k}'|\cos\theta',$$

where

$$M_1 = \mu^2 + \mu'^2 - \mu_1^2 . \tag{12}$$

So the integration over φ is trivial and we are led to the integral

$$\int_1^{-1} \frac{\mathrm{d}\cos\theta'}{((k-k')^2-\mu^2)^2} = \frac{2\sqrt{k_0^2-\mu^2}\,\sqrt{k_0'^2-\mu'^2}}{(M_1-2k_0k_0')^2-4(k_0^2-\mu^2)(k_0'^2-\mu'^2)} . \tag{13}$$

This is a function of w^2 and Δ^2 only, Δ^2 being contained in k_0 only, linearly (see formulae (6) and (8)). The function is singular when the denominator vanishes. This leads to the equation of second degree for k_0

$$4\mu'^2 k_0^2 - 4k_0' M_1 k_0 - (4\mu^2\mu'^2 - M_1^2) + 4\mu^2 k_0'^2 = 0 . \tag{14}$$

The descriminant of the equation is

$$D = -16(k_0'^2 - \mu'^2)(4\mu^2\mu'^2 - M_1^2). \tag{15}$$

It is:

$$4\mu^2\mu'^2 - M_1^2 = (2\mu\mu' + M_1)(2\mu\mu' - M_1) = ((\mu+\mu')^2 - \mu_1^2)(\mu_1^2 - (\mu-\mu)^2) \geqslant 0 \tag{16}$$

provided the triangular inequalities between the masses μ, μ', μ_1 are satisfied.

So, under this assumption, the singularities always occur at complex values of Δ^2, except for $\boldsymbol{k}' = 0$ ($w = \mu' + \mu''$): in this case they coincide on the real axis.

We compare now the position of these singularities with the position of the singularity $1/((p-p')^2-\mu_0^2)^2$. We start from the particular case $w = \mu' + \mu''$: from (6) and (14) we obtain

$$(p-p')^2 = \frac{\mu''(\mu_1^2 + \mu'^2 - \mu^2) + \mu'(\mu''^2 + \mu_1^2)}{\mu'} . \tag{17}$$

Then the condition that the singularity (17) occurs between the physical region and the pole at $(p-p')^2 = \mu_0^2$ is

$$\mu'(\mu''^2 + \mu_1^2 - \mu_0^2) + \mu''(\mu_1^2 + \mu'^2 - \mu^2) < 0 . \tag{18}$$

Considering the triangles constructed with the masses μ'', μ_1, μ_0 and μ', μ_1, μ respectively (see Fig. 2) we have

$$\begin{cases} \mu''^2 + \mu_1^2 - \mu_0^2 = 2\mu''\mu_1 \cos(\mu''\mu_1) , \\ \mu'^2 + \mu_1^2 - \mu^2 = 2\mu'\mu_1 \cos(\mu'\mu_1) , \end{cases} \tag{19}$$

so that (18) becomes

$$\cos(\mu'\mu_1) + \cos(\mu''\mu_1) < 0\,. \tag{20}$$

This condition is equivalent to the requirement that the angle between the sides μ' and μ'' of the quadrangle of Fig. 2 be larger than π. To this purpose it is in particular necessary that at least one of the angles between the sides μ', μ_1 and μ'', μ_1 be obtuse. This may happen only in the case of strange particles or nuclei.

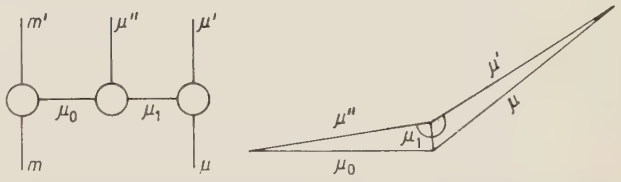


Fig. 2. – At the left: definition of the masses in the typical contribution that leads to the singularities studied here. At the right: the position of the singularities of the Δ^2 plane due to the intermediate mass μ_1 depends essentially from the angles in the quadrangle drown here with the masses: if the angle between the sides μ' and μ'' is larger then π then the singularities at threshold lie between the physical range and the pole at $(p-p')^2=\mu_0^2$. The quadrangle drawn here corresponds to the processes (d) or (d') of Fig. 3.

For increasing w the singularities describe in any case an hyperbola whose axis is the real Δ^2 axis. It is easy to see that the ascissa increases or decreases according to whether the angle $(\mu'\mu_1)$ (see Fig. 2) is smaller or larger then $\pi/2$.

Some examples and consequences will be discussed later.

Let us now briefly discuss the integrals of the type I_2 (formula 10). Let the function T be regular in the whole complex Δ^2 plane (this happens in the first non-vanishing order of perturbation theory). Then the integral I_2 is regular except for a cut connecting the two singular points of the already discussed integral I: these points are the roots of equation (14). It may be shown that this cut may be shifted, so that I_2 is regular except for two cuts connecting the above mentioned roots of equation (14) with infinity and one further cut connecting the point $\Delta^2=(m+\mu)^2$ with infinity (this last cut may be physically interpreted: it would correspond to a reaction where the particles of masses m and m' give rise to two particles of masses μ' and μ'', and total mass w, and one particle of mass μ). If T_1 is not everywhere regular (anyhow it cannot have poles), we in general expect I_2 to have eventually even more cuts but still no pole.

In any case the only assumption we actually use about the contributions of the type of I_2 is that their singularities do not destroy the poles due to the contribution I_1. As we have already stated this is proved in the first non-vanishing approximation of perturbation theory.

Finally the integral I_3 (formula (10)), may be easily transformed into the sum of two integrals of the type I_2 with a regular function T_1, so that it does not lead to new kinds of singularities.

3. – Discussion.

The results of the last section show that in general there are to expect singularities of the production cross section $\partial^2\sigma/\partial w^2\partial\Delta^2$ laying between the range of the physical values of the momentum transfer and the pole at $(p-p')^2=\mu_0^2$. (Such a possibility is not in disagreement with the analyticity within an ellipse proved by A. MINGUZZI and myself [2]). In fact only in particular cases the ellipse reaches the pole at $(p-p')^2=\mu_0^2$).

We give in Fig. 3 some examples of this kind, for which the inequality (20) is satisfied.

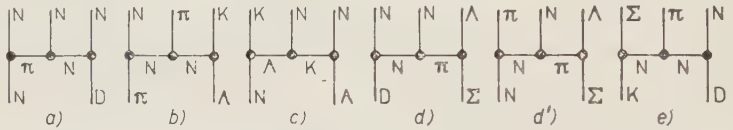


Fig. 3. – Examples of processes for which « anomalous poles » occur.

In these cases, as in many others, when w has its minimum value $\mu'+\mu''$, there is at least one pole on the real Δ^2 axis, between the physical range and the pole at $(p-p')^2=\mu_0^2$. So it is practically impossible to extrapolate the production cross section up to the pole at $(p-p')^2=\mu_0^2$. Such an extrapolation could be used, at least in principle, to get the cross section below threshold for the collision process where the particles of masses μ_0 and μ lead to the particles of masses μ' and μ'' [3]. (We remember that the angle between the sides μ' and μ'' of the quadrangle of Fig. 2 is larger then π: then it easily seen that the collision process under question is always exoergic. So the physical range of w begins at $w=\mu+\mu_0>\mu'+\mu''$.)

For increasing w the ascissa of the singularities in the Δ^2 plane decreases in all the examples of Fig. 3, because the angle between the sides μ_1 and μ' in Fig. 2 is larger then $\pi/2$. It follows that in some cases the Chew-Low extrapolation procedure cannot pratically be applied even to determine the

---

[2] R. ASCOLI and A. MINGUZZI: *Phys. Rev.*, **118**, 1435 (1960).

[3] The cross section for anelastic processes below threshold would have some interest in connection with the dispersion relations for such processes (private communication of A. MINGUZZI).

cross section above threshold for the above discussed collision process of the particles of masses μ_0 and μ. This happens for instance for the process (d') of Fig. 3. The positions of the expected singularities and the physical range are represented in this case in Fig. 4 for the threshold energy $w = \mu + \mu_0$

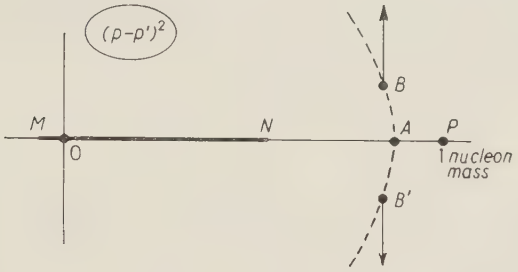


Fig. 4. – The complex $(p - p')^2$ plane in the case of process (d') of Fig. 3. P represents the usual pole at $(p - p')^2 = \mu_0^2$, μ_0 being here the nucleon mass. A is the pole due to the intermediate pion mass μ_1, when the total mass w of the outgoing $\mathcal{N}$ and Λ particles has its minimum values. When w increases, the singularities move along the dashed hyperbola (independently from W). B and B' are the poles in question when w has the threshold value of the energy of the reaction $\mathcal{N}+\Sigma \to \mathcal{N}+\Lambda$. M, N is the physical range in this case for $W = 175$ MeV above threshold. In this case the production cross section cannot be extrapolated up to the pole P neither using partial wave expansion nor using power series expansion.

of the process $\mathcal{N}+\Sigma \to \mathcal{N}+\Lambda$: the extrapolation cannot be done neither using one power series expansion nor the partial waves expansion.

In most of the examples of Fig. 3 the mass μ_0 leading to the pole at $(p - p')^2 = \mu_0^2$, is large. This mass may be the pion mass only where vertices with nuclei are considered. From this point of view case (a) of Fig. 3

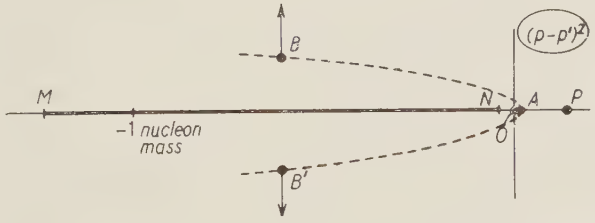


Fig. 5. – The complex $(p - p')^2$ plane in the case of process (a) of Fig. 3. P is the pole at $(p - p')^2 = \mu_0^2$, where μ_0 is here the pion mass. A is the pole due to the intermediate nucleon mass μ_1 when w is twice the nucleon mass. For increasing w the singularities describe the dashed hyperbola. They occur at B and B' when w has the threshold value of the reaction $\pi+\mathrm{D} \to \mathcal{N}+\mathcal{N}$. MN is the physical region in this case at $W = 145$ MeV. The extrapolation up to P is not possible using partial wave expansion. The situation is even worse, as W increases. In this example the « anomalous poles » are very near to the physical range.

is particulary interesting. In this case the singularities studied here occur, for $w = \mu' + \mu''$, at about $(p - p')^2 = 4Bm_{\mathcal{N}} = 0{,}0094\, m^2_{\mathcal{N}}$ where B is the binding energy of deuteron and $m_{\mathcal{N}}$ is the nucleon mass. So the singularities for some values of w and W (see Fig. 5) lay very near to the physical range of Δ^2. Then it may be expected that the behaviour of the cross section is dominated from these singularities (4). Here we do not investigate further this point.

4. – Conclusions.

Let us summarize the mean results of this kinetic investigation.

We have considered the production cross section $\partial^2\sigma/\partial w^2 \partial \Delta^2$ introduced by Chew and Low as function of the momentum transfer $\Delta^2 = -(p - p')^2/4$ at fixed W^2 and w^2. Besides the pole at $(p - p')^2 = m_0^2$ there are to expect, further singularities due to one-particle states: we have investigated the position of such singularities, particularly in connection with the Chew-Low extrapolation procedure.

The first result is that these new singularities are in general in the complex Δ^2 plane. They consist of poles at complex conjugate points and cuts extending from these points up to infinity. For the treshold value of w^2 the poles coincide on the real Δ^2 axis.

A more interesting phenomenon is that the abscissae of these singularities may be smaller then the abscissa of the pole occuring at $(p - p')^2 = \mu_0^2$, conjectured in particular by Chew and Low (1). This phoenomenon (which evidently occurs already for the amplitude itself) has some analogy with the « anomalous thresholds » found by Karplus, Sommerfield and Wichmann (5) in the scattering processes. Of course here the phenomenon is even simpler because in the case of production it is sufficient in perturbation theory, to consider Feynman graphs without integrations. Also here the occurence of the phenomenon is connected with some inequality between the masses (see formula (20) and Fig. 2); in particular one needs strange particles or nuclei; some examples are given Figs. 3, 4 and 5.

A consequence, rather academic at this moment, is that, in some cases, the Chew-Low extrapolation procedure to get scattering cross sections (1),

---

(4) Of course these singularities are due to the one particle states of mass μ_1. Here μ_1 is the nucleon mass, so we expect that the singularities due to two particle states (one nucleon an one meson) will also be very near to the physical range of Δ^2. (This remark is due to A. Minguzzi).

(5) R. Karplus, C. M. Sommerfield and E. H. Wichmann: *Phys. Rev.*, **114**, 376 (1959).

cannot be pratically applied; in fact before reaching the pole at $(p - p')^2 = \mu_0^2$ one may encounter these « anomalous poles » and cuts: an example is given in Fig. 4.

Finally in some cases (Figs 3 (*a*) and 5) the « anomalous poles » are so near to the physical range that the cross section may be expected to be dominated by them for some ranges of the variables.

This work in its first stages was done at CERN, Geneva, in collaboration with A. Minguzzi, to which I address my thanks.

RIASSUNTO

Nel presente lavoro si è indagata la natura e la posizione delle singolarità, dovute a stati di una particella, presumibili nella sezione d'urto di produzione $\partial^2\sigma/\partial w^2\,\partial \Delta^2$ introdotta da Chew e Low, quando sia considerata come funzione dell'impulso trasmesso Δ^2. Si trova che in generale queste singolarità sono nel piano complesso di Δ^2. Inoltre si trova un fenomeno analogo a quello delle « soglie anomale » dei processi di diffusione. Il fenomeno avviene solo quando è verificata una particolare disuguaglianza tra le masse implicate (vedi Fig. 2); tale disuguaglianza può solo essere soddisfatta quando intervengono particelle strane o nuclei: alcuni esempi sono dati in Fig. 3. Questo fenomeno può in taluni casi impedire l'applicazione del processo di estrapolazione proposto da Chew e Low per ottenere sezioni d'urto di diffusione da opportuni esperimenti di produzione. Due esempi particolarmente interessanti sono illustrati nelle Fig. 4 e 5.

Analytic Properties of Production Amplitudes as Functions of Two Momentum Transfers.

R. Ascoli

Istituto di Fisica dell'Università - Torino
Istituto Nazionale di Fisica Nucleare - Sezione di Torino

(ricevuto il 21 Luglio 1960)

Summary. — We show that, starting from the Dyson representation, it is possible to prove analytic properties of production amplitudes as functions of two dynamical variables at the same time. As such it is in particular possible to choose two invariant momentum transfers; the remaining variables which are held fixed are only subject to the restriction of determining completely the final momenta and may as well be covariant variables. There is no limitation to the number of final particles; there is only some simplification in the analyticity properties in the case of simple production. The research of the region of analyticity leads to an elegant mathematical problem. The solution of this problem leads to a region of analyticity which is the same in any case and consists of a finite domain in the space of the two complex variables; the domain includes always the physical region.

1. – Introduction.

After the last work of S. Mandelstam (1) it becomes more and more evident that progress in the knowledge of the analytic properties of scattering amplitudes is connected with progress in the problem of the analytic properties of production amplitudes.

A step in this direction was done by A. Minguzzi and myself (2) in proving analyticity properties with respect to one momentum transfer starting from the Lehmann-Jost-Dyson representation.

(1) S. Mandelstam: *Nuovo Cimento*, **15**, 658 (1960).
(2) A. Minguzzi and R. Ascoli: *Phys. Rev.*, **118**, 1435 (1960).

In the present work it is proved that any production amplitude has analytic properties with respect to two dynamical variables at the same time, in particular two invariant momentum transfers. The starting point is also here the Lehmann-Jost-Dyson representation.

2. – Outline of the method.

Let us call p_1 and p_2 the momenta of the initial particles of masses m_1 and m_2 respectively, $p_3 \dots p_n$ the momenta of the final particles of masses $m_3 \dots m_n$. We disregard spins.

Let us define the T matrix by

$$\langle p_3, p_4 \dots p_n \text{ out} | p_1, p_2 \text{ in} \rangle = i\delta(p_1 + p_2 - p_3 - p_4 \dots - p_n) T \,. \tag{1}$$

Then T depends on $3n - 10$ dynamical variables and we obtain, using reduction formulae,

$$T = \int \mathrm{d}x \exp\left[i \frac{p_1 - p_2}{2} x \right] \langle p_3 p_4 \dots p_n \text{ out} | R'\left(\psi_1\left(-\frac{x}{2}\right), \psi_2\left(\frac{x}{2}\right) \right) | 0 \rangle \,, \tag{2}$$

where ψ_1 and ψ_2 are the fields of the initial particles and

$$R'(\psi_1(x)\psi_2(y)) = -i(\Box_x - m_1^2)(\Box_y - m_2^2)\, \theta(x - y) [\psi_1(x), \psi_2(y)] \,. \tag{3}$$

Here we use local commutativity and spectral condition and apply the Jost-Lehmann-Dyson theory, so that we obtain for T the Dyson representation

$$T = \int \mathrm{d}^4 u \int \frac{\mathrm{d}\varkappa^2 \varphi(u, \varkappa^2, p_3, p_4, \dots, p_n)}{((p_1 - p_2)/2 - u)^2 - \varkappa^2} \,. \tag{4}$$

The function φ is arbitrary apart from a support condition; the support is well known and depends only from the spectrum of the particles and from the sum of the momenta of the two states of the matrix element (2). This sum is $p_1 + p_2$, the same as in the scattering process with the same initial particles. So the integration in (4) may be limited to the same region used by LEHMANN [3] to treat the scattering problem. Here it is only important to remember that in the center of mass system this region is spherically symmetric in the space of the vector $\boldsymbol{u}$.

Let us choose as dynamical variables first of all the $r = 3n - 12$ variables $w_1, w_2 \dots w_r$, which are needed to fix the relative position of the final vectors $\boldsymbol{p}_3, \boldsymbol{p}_4 \dots \boldsymbol{p}_n$. For instance in the case of simple production ($n = 5$) we may choose the square of the total energy $(p_3 + p_4 + p_5)^2 = (p_1 + p_2)^2 = W^2$ and the two quantities $(p_3 + p_4)^2 = w_1^2$, and $(p_3 + p_5)^2 = w_2^2$.

Here it is convenient to work in the center of mass system. Let us introduce

---

[3] H. LEHMANN: *Nuovo Cimento*, **10**, 579 (1958).

in the three-dimensional space any system of polar coordinates β and α defined only with reference to the vectors $\boldsymbol{p}_3, \boldsymbol{p}_4 \dots \boldsymbol{p}_n$. For instance in the case of simple production we may choose as polar plane the plane of the vectors $\boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5$.

Then it is clear that the function $\varphi(u, \varkappa^2, p_3, p_4, \dots p_n)$ depends only on the quantities: $w_1, w_2, \dots w_r, u^2, u_0, \beta, \alpha, \varkappa^2$. Indeed φ may depend only on $\varkappa^2$, u^2, the scalar products of pairs of final momenta, the scalar products of u with any final momentum (apart from the dependence from the masses of the final particles). Now the variables $w_1, w_2 \dots w_r$ determine any scalar product between two of the four-vectors $p_3, p_4 \dots p_n$ and the length and the polar coordinates of anyone of the vectors $\boldsymbol{p}_3, \boldsymbol{p}_4, \dots \boldsymbol{p}_n$; u^2, u_0, α and β determine the length and the polar coordinates of the vector $\boldsymbol{u}$. So also any scalar product between u and any vector $p_3, p_4, \dots p_n$, may be expressed through the quantities we have listed. So φ may depend only on them.

In the particular case of simple production and polar plane π containing $\boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5$, φ depends only on $w_1, w_2, W, u^2, u_0, \sin\beta, \alpha, \varkappa^2$: this is true because in this case the scalar products of $\boldsymbol{u}$ with $\boldsymbol{p}_3$ or $\boldsymbol{p}_4$ or $\boldsymbol{p}_5$ depend only on the projection $|\boldsymbol{p}|\sin\beta$ of $\boldsymbol{u}$ on π. So β appears only in the odd form $\sin\beta$. This fact will be used later.

Let us go back to the general case. From the fact that we were able to choose the coordinate system in such a way that only the $r = 3n - 12$ dynamical variables $w_1, w_2, \dots w_r$ appear in the function φ, immediately follows the foundamental conclusion that the two remaining dynamical variables from which the amplitude depends must only appear in the denominator of the Dyson representation (4). So we know always the explicit dependence of the integrand from them and we may always derive analyticity properties of the production amplitude as function of them.

To study these analyticity properties we first transform (4) as follows. The denominator may be written (W is the total energy)

$$(5)\qquad \left(\frac{p_1 - p_2}{2} - u\right)^2 - \varkappa^2 =$$

$$= -\left(|\boldsymbol{p}_1|^2 + |\boldsymbol{u}|^2 + \varkappa^2 - (u_0 + (m_1^2 - m_2^2)/2W)^2 - 2\,|\boldsymbol{u}|\,|\boldsymbol{p}_1|\cos(\boldsymbol{u}, \boldsymbol{p}_1)\right),$$

so that the expression (4) may be transformed into

$$(6)\qquad T = \int_{x_0(W)}^{\infty} \mathrm{d}x \int_{1}^{-1} \mathrm{d}\cos\beta \int_{0}^{2\pi} \mathrm{d}\alpha\, \frac{\Phi(x, w_1, w_2, \dots w_r, \beta, \alpha)}{x - \cos(\boldsymbol{u}, \boldsymbol{p}_1)},$$

where

$$(7)\qquad \Phi(x, w_1, w_2 \dots w_r, \beta, \alpha) = -\frac{1}{2\,|\boldsymbol{p}_1|} \int \mathrm{d}u_0 \int u\,\mathrm{d}u \int \mathrm{d}\varkappa^2 \cdot$$

$$\cdot\, \delta\left(x - \frac{|\boldsymbol{p}_1|^2 + |\boldsymbol{u}|^2 + \varkappa^2 - (u_0 + (m_1^2 - m_2^2)/2W)^2}{2\,|\boldsymbol{p}_1|\,|\boldsymbol{u}|}\right) \varphi(w_1, w_2 \dots w_r, u^2, u_0, \beta, \alpha, \varkappa^2).$$

The range of the integration over x is found by remarking that, as already explained, the domain of the integrations in (7) is the same as in the scattering problem with the same initial particles. So the variable x ranges from a minimum value $x_0(W)$ up to infinity and the minimum $x_0(W)$ may be taken from the theory developed by Lehmann for the case of scattering, and actually coincides with the major halfaxis of the Lehmann ellipse (4).

Here we remember only that it is always $x_0 > 1$. From this it follows immediately that the domain in which we may derive for T analyticity properties with respect to the two dynamical variables not appearing in the function φ certainly contains the physical region of both variables. Indeed for physical values of them, the $\cos(\boldsymbol{u}, \boldsymbol{p}_1)$ is certainly less then 1, so that the denominator in (6) cannot vanish.

3. – Choice of the variables and kind of the deduced analyticity properties.

Let us now choose the two further dynamical variables on which T depends besides $w_1, w_2, \ldots w_r$. We have found convenient to choose the cartesian components X and Y of the unit vector of direction $\boldsymbol{p}_1$ on two orthogonal axes defined in some way by using only the final vectors $\boldsymbol{p}_3, \boldsymbol{p}_4, \ldots \boldsymbol{p}_n$ (5). To treat the problem we define the polar co-ordinates of $\boldsymbol{u}$ taking as polar plane π the plane of these two orthogonal axes and we count the azimuths starting from the axis of X.

Then we have for the denominator of (6)

$$x - \cos(\boldsymbol{u}, \boldsymbol{p}_1) = x - \sin\beta\cos\alpha X - \sin\beta\sin\alpha Y - \cos\beta\sqrt{1 - X^2 - Y^2}\,. \tag{8}$$

So we see that T is in general not an analytic function of X and Y also in the region where the denominator (8) does not vanish. This is due to the square root appearing in (8) (of course we have in the integrand the explicit dependence on X and Y, so we know the kind of singularity introduced by this square root and eventually we may look for uniformizing variables).

As already remarked, for any value of X and Y there are two different configurations of the vectors $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots \boldsymbol{p}_n$. They are obtained by reflection

---

(4) The quantity $x_0(W)$ has actually the same name in Lehmann's work (3) and is given by formula (12) of his work (of course our x differs from his own by a factor $\sin\beta$).

(5) Of course there are in general two values of the amplitude for any value of these variables. We shall see below that this is unimportant, because whenever it happens, also analyticity ceases to hold: in this case analyticity holds for combinations of the amplitudes which are completely determined by these variables.

of the vectors $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ with respect to the plane π. These two configurations lead in general to two different values of the amplitude corresponding to the two different signs before the square root of formula (8). If T is one of them we call T_π the other one.

Then we see easily that the combinations $T+T_\pi$ and $(T-T_\pi)/\sqrt{1-X^2-Y^2}$ are analytic functions of X and Y whenever the denominator (8) does not vanish. This is analogous to what has been found in our previous work with A Minguzzi [2]: from (6) and (8) we have

$$(9)\qquad T=\int_{x_0(W)}^{\infty}\mathrm{d}x\int_0^{2\pi}\mathrm{d}\alpha\cdot$$

$$\cdot\int_1^0\mathrm{d}\cos\beta\cdot\left(\frac{\Phi(x,w_1,w_2\dots w_r,\beta,\alpha)}{x-\sin\beta\cos\alpha X-\sin\beta\sin\alpha Y-\cos\beta\sqrt{1-X^2-Y^2}}+\right.$$

$$\left.+\frac{\Phi(x,w_1,w_2\dots w_r,\pi-\beta,\alpha)}{x-\sin\beta\cos\alpha X-\sin\beta\sin\alpha Y+\cos\beta\sqrt{1-X^2-Y^2}}\right),$$

$$(10)\qquad T_\pi=\int_{x_0(W)}^{\infty}\mathrm{d}x\int_0^{2\pi}\mathrm{d}\alpha\cdot$$

$$\cdot\int_1^0\mathrm{d}\cos\beta\left(\frac{\Phi(x,w_1,w_2\dots w_r,\beta,\alpha)}{x-\sin\beta\cos\alpha X-\sin\beta\sin\alpha Y+\cos\beta\sqrt{1-X^2-Y^2}}+\right.$$

$$\left.+\frac{\Phi(x,w_1,w_2\dots w_r,\pi-\beta,\alpha)}{x-\sin\beta\cos\alpha X-\sin\beta\sin\alpha Y-\cos\beta\sqrt{1-X^2-Y^2}}\right).$$

So it is

$$(11)\qquad T+T_\pi=\int_{x_0(W)}^{\infty}\mathrm{d}x\int_0^{2\pi}\mathrm{d}\alpha$$

$$\cdot\int_1^0\mathrm{d}\cos\beta\,\frac{2(x-\sin\beta\cos\alpha X-\sin\beta\sin\alpha Y)(\Phi(x,w_1,w_2\dots w_r,\beta,\alpha)+\Phi(x,w_1,w_2\dots w_r,\pi-\beta}{(x-\sin\beta\cos\alpha X-\sin\beta\sin\alpha Y)^2-\cos^2\beta(1-X^2-Y^2)}$$

$$(12)\qquad \frac{T-T_\pi}{\sqrt{1-X^2-Y^2}}=\int_{x_0(W)}^{\infty}\mathrm{d}x\int_0^{2\pi}\mathrm{d}\alpha\cdot$$

$$\cdot\int_1^0\mathrm{d}\cos\beta\cdot\frac{2\cos\beta(\Phi(x,w_1,w_2\dots w_r,\beta,\alpha)-\Phi(x,w_1,w_2\dots w_r,\pi-\beta,\alpha))}{(x-\sin\beta\cos\alpha X-\sin\beta\sin\alpha Y)^2-\cos^2\beta(1-X^2-Y^2)},$$

where the integrands are analytic functions of X and Y regular as long as the denominators do not vanish. So also the integrals are analytic at least in the region where the denominators do not vanish in the domain of the integrations.

There are of course special cases in which the amplitude T itself is analytic within the region where the denominator does not vanish in the domain of integration. As in the case of the work ([2]) the necessary and sufficient condition for this to happen is $T = T_\pi$.

A particularly important case of this type is the case of simple production, provided we choose π to be the plane of the momenta $\boldsymbol{p}_3$, $\boldsymbol{p}_4$, $\boldsymbol{p}_5$ of the produced particles. Then, as already remarked, the weight function of the representation depends on β only through $\sin\beta$, so that from (7) and (12) it follows immediately that $T = T_\pi$. This is particularly interesting because, as we shall see later, with such a choice of the π plane the variables X and Y are linearly connected with two invariant momentum transfers.

4. – Research of the region of analyticity.

To solve completely the problem we have now to find the region of the space of the two variables X and Y where the denominator of (11) or (12) does not vanish for any value of α between 0 and 2π and any value of β between 0 and π. This is a very elegant mathematical problem.

In fact the problem is invariant against the group of three-dimensional rotations (before choosing the system of coordinates X and Y). This is clear when the denominator is written like in (6): $x - \cos(\boldsymbol{u}, \boldsymbol{p}_1)$, where $\boldsymbol{u}$ may take any direction.

Therefore, when the coordinates X and Y are chosen the result must be invariant against rotations of the axes X and Y in the three-dimensional space. This requirement will be sufficient to determine completely the region of analyticity apart from the value of one parameter.

To get the result in a simple way we remark firstly that the real parts X_r and Y_r and the imaginary parts X_i and Y_i of X and Y define respectively two vector of components X_r, Y_r and X_i, Y_i in the plane π. We call ϱ_r and ϱ_i their lengths and φ their relative angle (see Fig. 1); we have

$$(13) \qquad \varrho_r^2 = X_r^2 + Y_r^2, \qquad \varrho_i^2 = X_i^2 + Y_i^2, \qquad \varrho_r \varrho_i \cos\varphi = X_r X_i + Y_r Y_i .$$

Invariance against rotations in the π plane implies that the result must have the form of a relation among ϱ_r, ϱ_i and φ.

Now we have to use the arbitrariness in the choice of the coordinate system. To this purpose we consider a new coordinate system with axes X', Y'

and we call σ their plane. We assume that the system X', Y' is obtained from the system X, Y through a rotation by an angle ζ around the X axis followed by a rotation by an angle ψ around the Z' axis orthogonal to X' and Y' (see Fig. 1). (We need not consider the third Euler angle because we have already taken into account the symmetry against rotations in the π plane).

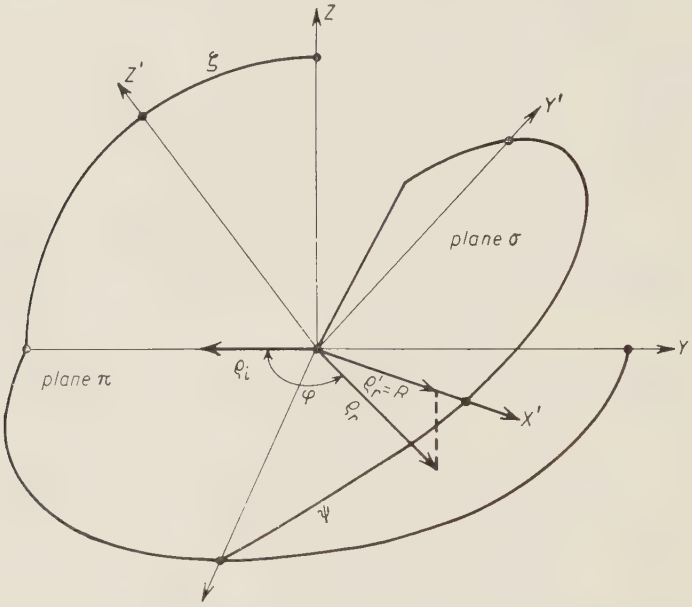


Fig. 1. – In the plane π are represented the real and imaginary parts of the projection of the unit vector with direction $\boldsymbol{p}_1$ on the plane π. Their lengths are ϱ_r, ϱ_i, their relative angle φ. On the arbitrary plane σ particular fixed values of the corresponding quantities ϱ'_r, ϱ'_i, φ' have been chosen: $\varrho'_i = 0$, $\varrho'_r = R$. The rotational symmetry of the problem implies that the set of values of ϱ_r, ϱ_i and φ, obtained by letting the plane σ vary, describes the boundary of the domain of analyticity. In this way eq. (20) or (21) are obtained.

Then we consider in the $X'Y'$ system one particularly simple set of values of X' and Y': $X'_i = 0$, $Y' = 0$. We call R the corresponding value of X_r that solves the problem. The third component Z' of the unit vector with direction $\boldsymbol{p}_1$ will be $Z' = \sqrt{1 - R^2}$. As we have already seen any point of the boundary of the analyticity domain corresponds to complex vectors $\boldsymbol{p}_1$: so we are interested only in the case $R > 1$ (the excluded case leads simply to a circle of radius 1 in the real plane). Then $Z' = i\sqrt{R^2 - 1}$ is imaginary.

Now we consider the components X and Y of the particular unit vector considered above. Owing to the rotational symmetry of the problem, the values of X and Y so obtained must also solve the problem for any position

of the system X', Y' with respect to the system X, Y. By letting the angles ζ and ψ vary, we must obtain in this way all the possible sets of values of ϱ_r, ϱ_i and φ (defined by eq. (13)).

We have from Fig. 1 (for $Y' = 0$)

$$\begin{cases} X = X' \cos\psi \\ Y = X' \sin\psi \cos\zeta + Z' \sin\zeta \,. \end{cases} \tag{14}$$

So it is, introducing the values $X' = R$, $Z' = i\sqrt{R^2 - 1}$,

$$\begin{cases} X_r = R\cos\psi \,, & X_i = 0 \,, \\ Y_r = R\sin\psi\cos\zeta \,, & Y_i = \sqrt{R^2-1}\sin\zeta \,. \end{cases} \tag{15}$$

For $X_i = 0$ we have from (13) $Y_i = \varrho_i$, $Y_r = \varrho_r \cos\varphi$, $X_r = \varrho_r \sin\varphi$. So, in terms of ϱ_r, ϱ_i and φ, (15) becomes

$$\begin{cases} \varrho_r \sin\varphi = R\cos\psi \,, \\ \varrho_r \cos\varphi = R\sin\psi\cos\zeta \,, & \varrho_i = \sqrt{R^2-1}\sin\zeta \,. \end{cases} \tag{16}$$

By eliminating the angles ζ and ψ we have the desired relation between ϱ_r, ϱ_i and φ

$$\varrho_r^2(R^2-1) + \varrho_i^2 R^2 - \varrho_r^2\varrho_i^2 \sin^2\varphi - R^2(R^2-1) = 0 \,. \tag{17}$$

Therefore the requirement of rotational symmetry determines completely the region of analyticity up to the constant R (6).

The value of the constant R may now be easily derived considering one particular case in our problem concerning the zeros of the denominator of (11) or (12).

We consider for instance again the particular case $Y = 0$, $X_i = 0$. Then we obtain, by equating the denominator of (11) or (12) to zero and, using the notation (13),

$$(1 - \varrho_r^2)\cos^2\beta = (x - \varrho_r \sin\beta\cos\alpha)^2 \,. \tag{18}$$

(6) The consideration of functions analytic within a domain which is invariant against changes of variables due to rotations is equivalent to the introduction of the concept of an analytic function of one complex direction in the three-dimensional space. This concept is a generalisation of the concept of analytic function of one complex direction in a plane, implicit in the derivation of the Lehmann ellipse: in fact the Lehmann ellipse is invariant for rotations in the scattering plane, and may be derived from this property apart from one constant parameter.

Then we have to look for the minimum value of ϱ_r satisfying this equation when α and β vary within their ranges. This minimum value is R, by definition.

The eq. (18) cannot be solved with respect to ϱ_r (ϱ_r must be real) except in the case $\cos\beta = 0$. In this case the eq. (18) gives immediately

$$\varrho_r = \frac{x}{\cos\alpha}. \tag{19}$$

So the minimum value of ϱ_r occurs for $\cos\alpha = 1$ and is equal to x. So we obtain $R = x$.

The region (17) clearly increases in any direction by increasing x. So the region where the denominator of the integrands of (11) or (12) does never vanish during the integration corresponds to the minimum value $x_0 = x_0(W)$ of x (see footnote (4)).

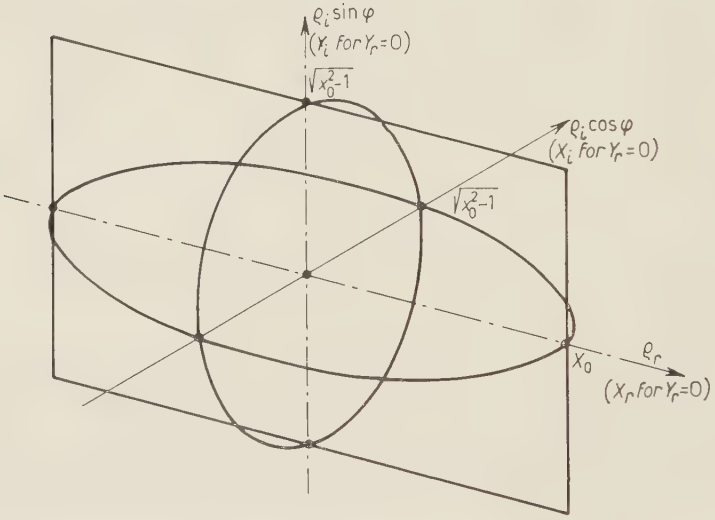


Fig. 2. – The three-dimensional surface given by eq. (20) or (21). In the case $Y_r = 0$ the axes represent simply X_r, X_i and Y_i. In general they represent

$$\varrho_r = \sqrt{X_r^2 + Y_r^2}, \qquad \varrho_i \cos\varphi = \frac{X_r X_i + Y_r Y_i}{\sqrt{X_r^2 + Y_r^2}} \qquad \varrho_i \sin\varphi = \frac{X_r Y_i - X_i Y_r}{\sqrt{X_r^2 + Y_r^2}}.$$

The intersection of the surface with the plane $\varrho_i \sin\varphi = 0$ is an ellipse, the intersection with the plane $\varrho_i \cos\varphi = 0$ is a rectangle. The region of analyticity in the four dimensional space of X_r, X_i, Y_r, Y_i is obtained by rotating by the same angle the vectors X_r, Y_r and X_i, Y_i in the π plane (see Fig. 1).

So finally the boundary of the region of analyticity is given by the equation

$$\varrho_r^2(x_0^2 - 1) + \varrho_i^2 x_0^2 - \varrho_r^2\varrho_i^2 \sin^2\varphi - x_0^2(x_0^2 - 1) = 0. \tag{20}$$

Going back to the variables X_r, X_i, Y_r, Y_i we have

$$\frac{X_r^2+Y_r^2}{x_0^2}+\frac{X_i^2+Y_i^2}{x_0^2-1}-\frac{(X_rY_i-X_iY_r)^2}{x_0^2(x_0^2-1)}=1\,. \tag{21}$$

This surface is represented in Fig. 2 with suitable coordinates.

5. – Connection of the variables X and Y with the invariant momentum transfers.

One of the advantages of the variables X and Y is that they are linearly connected with two invariant momentum transfers.

Indeed let us choose for convenience as plane π the plane of two final momenta, $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$ for instance (in the c.m. system). Let us call 2χ the angle between them. The angle 2χ is determined by the variables $w_1, w_2, \ldots w_r$ which fix the configuration of the final vectors (the expression of $\cos 2\chi$ as function of $w_1, w_2, \ldots w_r$ is particularly simple when $w_1, w_2, \ldots w_r$ are covariant variables like products of the final four-vectors).

Then we choose in the plane π the bisectrix of the angle 2χ as direction of X.

Let us call b and a the polar coordinates of $\boldsymbol{p}_1$, when π is the polar plane and the azimuths a are counted starting from the direction of X (see Fig. 3).

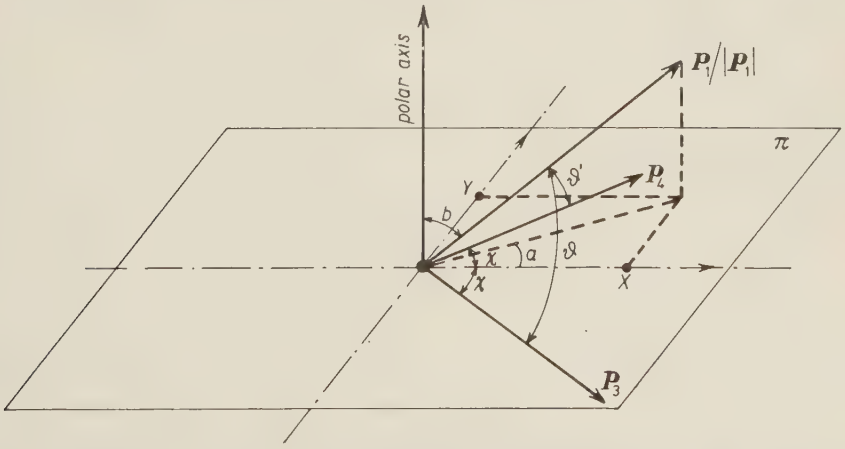


Fig. 3. – Connection of the two invariant momentum transfers $(p_1-p_3)^2$ and (p_1-p_4), with the projections X and Y of the unit vector of direction $\boldsymbol{p}_1$, on the plane π of $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$; θ and θ' are the angles between $\boldsymbol{p}_1$, $\boldsymbol{p}_3$ and $\boldsymbol{p}_1$, $\boldsymbol{p}_4$ respectively. $\cos\theta$ and $\cos\theta'$ are linearly connected to the momentum transfers and to X and Y.

Let us now consider as dynamical variables which fix the direction of $\boldsymbol{p}_1$ the two invariant momentum transfers $(p_1-p_3)^2$ and $(p_1-p_4)^2$. These two quantities are linearly connected with $\cos\theta$ and $\cos\theta'$, where θ and θ' are respectively the angles of $\boldsymbol{p}_3$ and $\boldsymbol{p}_4$ with $\boldsymbol{p}_1$ in the c.m. system.

Then we have (see Fig. 3)

$$(22) \qquad \begin{cases} \cos\theta = \sin b \cos(a + \chi) \\ \cos\theta' = \sin b \cos(a - \chi) . \end{cases}$$

We use

$$(23) \qquad \begin{cases} X = \sin b \cos a \\ Y = \sin b \sin a . \end{cases}$$

Then by summing and subtracting the eq. (22), and dividing the results by $2\cos\chi$ and $2\sin\chi$ respectively we have, using (23),

$$(24) \qquad X = \frac{\cos\theta + \cos\theta'}{2\cos\chi}, \qquad Y = \frac{\cos\theta - \cos\theta'}{2\sin\chi},$$

which proves the linear connection stated above.

The result (21) may be easily expressed in term of the variables $\cos\theta$ and $\cos\theta'$.

By decomposing $\cos\theta$ and $\cos\theta'$ into real and imaginary part,

$$(25) \qquad \cos\theta = \mathscr{R} + i\mathscr{I}, \qquad \cos\theta' = \mathscr{R}' + i\mathscr{I}',$$

we have from (21) and (24)

$$(26) \qquad \frac{\mathscr{R}^2 + \mathscr{R}'^2 - 2\mathscr{R}\mathscr{R}'\cos 2\chi}{x_0^2} + \frac{\mathscr{I}^2 + \mathscr{I}'^2 - 2\mathscr{I}\mathscr{I}'\cos 2\chi}{x_0^2 - 1} - \\ - \frac{(\mathscr{R}\mathscr{I}' - \mathscr{R}'\mathscr{I})^2}{x_0^2(x_0^2 - 1)} = \sin^2 2\chi .$$

The combinations of amplitudes (11) and (12) become now

$$T + T_\pi \quad \text{and} \quad (T - T_\pi)\sin 2\chi / \sqrt{\sin^2 2\chi - \cos^2\theta - \cos^2\theta' + 2\cos\theta\cos\theta'\cos 2\chi} .$$

In the particular case of simple production we have shown that the amplitude T itself has analyticity properties with respect to the variables X and Y, provided the π plane coincides with the plane of the final particles. From the work of this section it follows immediately that in the case of simple production we have analyticity properties of the amplitude T also with respect to a pair of invariant momentum transfers, $(p_1 - p_3)^2$ and $(p_1 - p_4)^2$ for instance.

6. – Discussion of the results.

a) Let us now first discuss the results in terms of the variables X and Y.

We consider firstly the section of the region (21) with the plane of the real parts of the two variables X and Y ($X_i = 0$, $Y_i = 0$). This section is clearly a circle C of radius x_0 and centrum in the origin of the plane of X_r and Y_r. This result is to be compared with the physical region in the same plane, which is a circle with the same centrum and radius 1 (see Fig. 4*a*).

For any point of the circle C of radius x_0 we have now to ask which is the region of analyticity in the plane of the variables X_i and Y_i.

Firstly we remember that there is a symmetry against simultaneous rotations of the same angle in the two planes X_r, Y_r and X_i, Y_i. So it is easy to obtain the region of analyticity in the plane X_i, Y_i, for any arbitrary point of the circle C of the plane X_r, Y_r, provided we know the region for the points of a radius, for instance $Y_r = 0$: we have simply to rotate the region in the plane X_i, Y_i by the azimuth of the arbitrary point counted from the X_r axis.

We examine then the behaviour of the region of analyticity in the X_i, Y_i plane as function of X_r for $Y_r = 0$. This is particularly clear from Fig. 2.

For the centrum of the circle C, that is for $X_r = 0$, $Y_r = 0$, we have in the X_i, Y_i plane a circle of radius $\sqrt{x_0^2 - 1}$ and centrum at the origin (Fig. 4*b*).

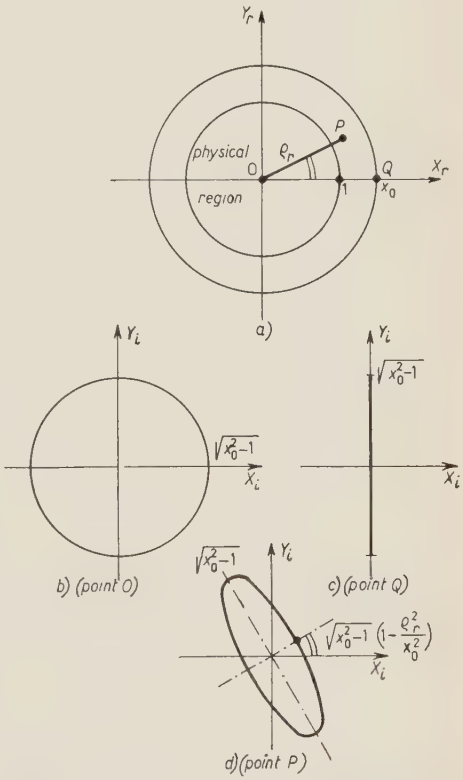


Fig. 4. – *a*) The plane of the real values of the variables X and Y: the physical region and the region of analyticity (circle of radius x_0). *b-c-d*) The region of analyticity in the plane of the imaginary values of X and Y for the points O, Q and P of Fig. 4*a*.

By letting X_r vary at $Y_r = 0$ the region in the X_i, Y_i plane is an ellypse with axes in the X_i and Y_i directions. The major halfaxis is constant and equal to $\sqrt{x_0^2 - 1}$, because for $Y_r = 0$, $X_i = 0$ (21) represents a rectangle in in the plane X_r, Y_i. On the contrary the minor halfaxis $\sqrt{x_0^2 - 1}(1 - (X_r^2/x_0^2))$ decreases for increasing X_r and becomes zero at $X_r = x_0$ (see Fig. 4*c*).

In fact for $Y_r = 0$, $Y_i = 0$ (21) represents (see Fig. 5) an ellipse in the plane X_r, X_i (equal to the Lehmann ellipse of the scattering problem with the same initial particles).

As explained before the analyticity region in the X_i, Y_i plane has simply to be rotated around its centrum when $Y_r \neq 0$ (see Fig. 4*d*).

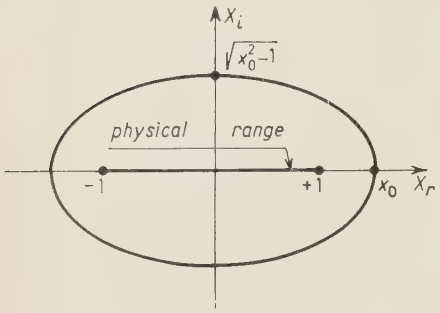


Fig. 5. – The region of analyticity in the plane of the complex variable X for $Y = 0$.

b) Let us now go over to the variables $\cos\theta$ and $\cos\theta'$. We introduce firstly the variables

$$(27) \qquad \begin{cases} X' = \dfrac{\cos\theta + \cos\theta'}{\sqrt{2}}, \\ Y' = \dfrac{\cos\theta - \cos\theta'}{\sqrt{2}}. \end{cases}$$

These variables are obtained from $\cos\theta$, $\cos\theta'$ by a rotation by 45° around the origin in the $\cos\theta$, $\cos\theta'$ plane.

Then we have

$$(28) \qquad X = X'/\sqrt{2}\cos\chi\,, \qquad Y = Y'/\sqrt{2}\sin\chi\,.$$

As before we examine first the section of the analyticity region with the plane of the real parts $\mathscr{R}$ and $\mathscr{R}'$ of $\cos\theta$ and $\cos\theta'$.

This is clearly the ellipse E

$$\left(\frac{X'}{\sqrt{2}\cos\chi}\right)^2 + \left(\frac{Y'}{\sqrt{2}\sin\chi}\right)^2 = x_0^2\,.$$

The ellipse is represented in Fig. 6. The halfaxes are $x_0\sqrt{2}\cos\chi$ and $x_0\sqrt{2}\sin\chi$, and form an angle of 45° with the $\mathscr{R}$ and $\mathscr{R}'$ axes. The ellipse is inscribed in the square of side x_0 ($\mathscr{R} = \pm x_0$, $\mathscr{R}' = \pm x_0$). This ellipse has to be compared with the physical region of $\cos\theta$ and $\cos\theta'$. The boundary of this region is obtained from (22) by putting $\sin b = 1$. Then (22) are the parametric equations of an ellipse inscribed in the square of side 1 and center at the origin: $\mathscr{R} = \pm 1$, $\mathscr{R}' = \pm 1$ (see Fig. 6).

The axes lie on the diagonals and their halflengths are $\sqrt{2}\cos\chi$ and $\sqrt{2}\sin\chi$. For $2\chi = 0$ or $2\chi = \pi$ the ellipse reduces to a straight line (a diagonal of the square), whereas for $2\chi = \pi/2$ it reduces to a circle. So the region of analyticity is homothetic to the physical region with respect to the origin in the $\mathscr{R}$, $\mathscr{R}'$ plane.

Let us now discuss briefly the region of analyticity in the plane of the imaginary parts $\mathscr{I}$ and $\mathscr{I}'$ of $\cos\theta$ and $\cos\theta'$, for any point of the ellispe E of the plane $\mathscr{R}$, $\mathscr{R}'$.

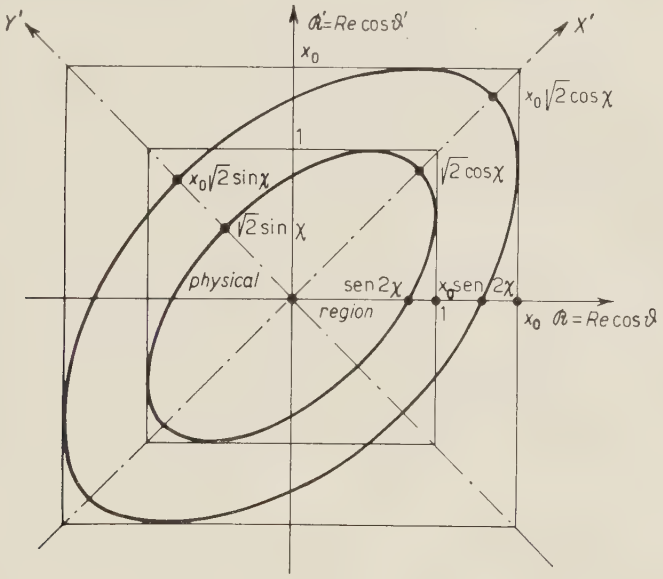


Fig. 6. – The plane of the real values of the variables $\cos\theta$ and $\cos\theta'$: the physical region and the region of analyticity (larger ellipse). θ and θ' are the angles between one initial momentum and two final momenta respectively, 2χ is the angle between these two final momenta. x_0 is the major halfaxis of the Lehmann ellipse.

For the centrum of this ellipse $\mathscr{R}=0$, $\mathscr{R}'=0$ the region in the $\mathscr{I}$, $\mathscr{I}'$ plane is an ellipse similar to the ellipse E, but smaller according to the ratio $\sqrt{x_0^2-1}/x_0$. By letting now $\mathscr{R}$ vary at $\mathscr{R}'=0$, the region in the plane $\mathscr{I}$, $\mathscr{I}'$ is an ellipse which cuts the $\mathscr{I}$ and $\mathscr{I}'$ axes at points which approach the origin as $\mathscr{R}$ approaches its maximum value $x_0 \sin 2\chi$. For this value of $\mathscr{R}$ the region becomes a straight line with direction conjugated with respect to the ellipse E to the direction of the $\mathscr{R}$ axis. The points where the ellipse in the $\mathscr{I}$, $\mathscr{I}'$ plane cuts the $\mathscr{I}$ axis correspond to $\cos\theta'=0$ and are given by the ellipse in the $\cos\theta$ plane represented in Fig. 7.

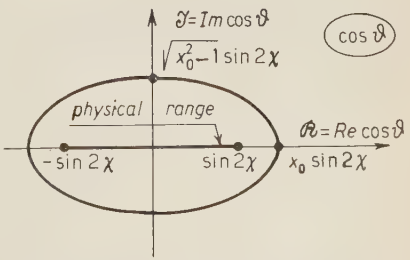


Fig. 7. – The region of analyticity in the plane of the complex variable $\cos\theta$ for $\cos\theta'=0$.

7. – Conclusions.

In this work we have shown that, starting from the Dyson representation, it is possible to derive analytic properties of production amplitudes with respect to two dynamical variables at the same time.

In this treatment we have left from the beginning an arbitrariness in the choice of these two dynamical variables, as explained in Sect. **3**. It is of course interesting that it is possible to choose for them two invariant momentum transfers. The remaining dynamical variables, which are kept fixed, are only subject to the restriction of determining completely the final momenta and may as well be covariant variables.

With the variables we have used, the amplitude itself has not in general simple analytic properties. The simplest quantities that have simple analytic properties with respect to these variables are combinations of the values of the amplitude corresponding to two configurations of the momenta in the center of mass system (see Sect. **3**) (7).

There is of course an important case where the amplitude itself has simple analytic properties. This is the case of simple production considered as function of two invariant momentum transfers.

Then we have investigated the analyticity region. This is an elegant mathematical problem and we have soved it by considering the fact that it is invariant against the group of rotations in three-dimensional space. The region found in this way is finite and includes in any case the physical region. In the last section we have discussed some particular features of this region that are illustrated in the figures 4, 5, 6, 7: it is particularly interesting to look at the domain of analyticity in the plane of the real parts of the variables (see Fig. 4 and 6). Taking as variables two momentum transfers, this region is an ellipse homothetic, with respect to the centrum, to the ellipse which limits the physical region. The homothety ratio is equal to the larger halfaxis x_0 of the Lehmann ellipse.

The method used here opens the way to many similar investigations on production amplitudes. There is no difficulty in deriving from the Dyson representation analytic properties with respect to other pairs of dynamical variables, starting with a different use of the reduction formulae (one of the variables may even be the energy). The investigation may also be extended to amplitudes with more then two ingoing lines (8).

---

(7) This result is analogous to the one found in our previous work with A. Minguzzi.

(8) Even the investigation of such amplitudes will be important in the future, because they enter always into the unitarity relation, when applied to production processes. The results of this paper may be applied also to such amplitudes with more than two ingoing lines, with a suitable choice of the dynamical variables.

We believe that these and similar investigations will enable a better knowledge of the production processes: such a knowledge seems at the moment to be fundamental for an extension of the region where the analytic properties of scattering amplitudes are rigorously proved ([1,9]). Besides it is also requested for improvements of the approximation methods used to calculate scattering amplitudes from their analyticity properties.

([9]) We have proved, with A. Bottino and A. Molinari, that the derivation by use of unitarity of analyticity properties of scattering amplitudes from analyticity properties of production amplitudes actually requires analyticity with respect to two momentum transfers, for any number of produced particles. Of course clearly the domain of analyticity that has been found in this work is not sufficiently large to enable an extension of the known analyticity properties of the imaginary part of the scattering amplitudes beyond the large Lehmann ellipse. To this purpose it would be necessary to extend the domain that has been found here with a method similar to the one used by Mandelstam ([1]) for the scattering. We do not know whether this is possible, but in any case this implies the derivation of a wider analyticity domain for the imaginary part of the production amplitude: this will be done in the near future with methods analogous to those introduced here.

RIASSUNTO

Si mostra che è possibile dedurre dalla rappresentazione di Dyson proprietà di analiticità delle ampiezze di produzione, considerate come funzioni di due variabili dinamiche allo stesso tempo. Si può in particolare scegliere come tali due impulsi trasmessi invarianti; le rimanenti variabili che sono mantenute costanti sono soggette alla sola condizione di determinare completamente la configurazione relativa degli impulsi finali e possono essere anche esse variabili invarianti. Non vi è limitazione alcuna per il numero di particelle finali. La ricerca della regione di analiticità conduce a un elegante problema matematico, che si risolve usando proprietà di simmetria. Si giunge così a un dominio di analiticità finito nello spazio delle due variabili complesse (eq. (21) e Figg. 2, 4, 5, 6, 7). Il dominio contiene in ogni caso la regione fisica.

Analytic Properties of Bound States in Potential Theory.

L. Bertocchi (*), C. Ceolin and M. Tonin

Istituto di Fisica dell'Università - Padova
Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 28 Luglio 1960)

Summary. — The analytic properties of the vertex functions of a composite particle (« deuteron ») are analyzed in a potential model. The results are compared with the « anomalous » singularities of the relativistic perturbation theory.

1. – Introduction.

One of the most interesting programs in dispersion theory is the application of the dispersion methods to the study of problems of nuclear physics.

The advantage of the dispersion approach on the conventional ones, lies in the fact that it allows to take into account in an appropriate manner the relativistic effects.

The main difficulty in such an application is due to the fact that the nuclear systems are loosely bound and relativistic perturbation theory predicts a location of the branch points, which is rather different from what one would guess from unitarity [1].

Such « anomalous » thresholds are certainly the main feature of the dispersion relations for nuclear problems, and therefore, a deeper understanding of their meaning is needed for a successful development of the dispersion program.

At present it has not been possible to go beyond lowest order perturbation theory and even understand completely the meaning of the use of Feynman

(*) Permanent address: Istituto di Fisica dell'Università, Bologna.

(1) R. Karplus, C. M. Sommerfield and E. H. Wichmann: *Phys. Rev.*, **111**, 1187 (1958).

graphs for nuclear problems, since we do not have any hamiltonian theory from which such graphs can be derived.

We think, therefore, that a comparison between the results obtained in relativistic perturbation theory and in the conventional approach, is very instructive. In this paper, we will show that also in the case of potential theory the vertex functions in which one or two deuteron lines enter can be represented in a spectral form.

We shall assume that the deuteron is an eigenstate of a Schrödinger equation with a potential of the form

$$V(r) = \int_{\mu_0}^{\infty} g'(\mu) \frac{\exp[-\mu r]}{r} \, \mathrm{d}\mu \,. \tag{1.1}$$

We know that for potentials of the type (1.1) it has been possible to derive a Mandelstam representation for nucleon-nucleon scattering [2].

The method used in order to derive the spectral representation is based on a Laplace expansion of the deuteron wave function; such a method has been used for scattering problem by MARTIN [3]. A corresponding form of the wave function in momentum space has been introduced by BLAKENBECLER and COOK [4].

For the vertex functions considered the potential method fully confirms the predictions of lowest order perturbation theory, and allows to obtain the analyt'c properties of all orders in perturbation theory, when a static approximation is taken.

2. – The method.

2·1. – Let us consider a system of 2 spinless particles (which we shall call « proton » and « neutron »), interacting through a central potential $V(r)$

$$\begin{cases} V(r) = \int_{\mu_0}^{\infty} g'(\mu) \frac{\exp[-\mu r]}{r} \, \mathrm{d}\mu = \int_{\mu_0}^{\infty} g(\mu) \exp[-\mu r] \, \mathrm{d}\mu \,, \\ g(\mu) = \int_{\mu_0}^{\infty} g'(\lambda) \theta(\mu - \lambda) \, \mathrm{d}\lambda \,. \end{cases} \tag{2.1}$$

---

[2] R. BLANKENBECLER, M. L. GOLDBERGER, N. N. KHURI and S. B. TREIMAN *Ann. Phys.*, **10**, 62 (1960).

[3] A. MARTIN: *Nuovo Cimento*, **14**, 403 (1959); **15**, 99 (1960).

[4] L. F. COOK and R. BLANKENBECLER: to be published.

We shall first study the case of a bound state of angular momentum $l = 0$.

The Schrödinger equation for the radial wave function $u(r) = r\psi(r)$ is

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \chi^2\right) u(r) = M u(r) \int_{\mu_0}^{\infty} g(\mu) \exp\left[-\mu r\right] \mathrm{d}\mu \,, \tag{2.2}$$

where $\chi^2 = MB$, M is the mass of the « nucleon » and B is the binding energy.

The solution of equation (2.2), corresponding to a bound state wave function must satisfy the following boundary conditions

$$u(r) = 0 \qquad \text{for } r = 0 \tag{2.3'}$$

$$u(r) \sim N \exp\left[-\chi r\right], \qquad \text{for } r \to \infty \tag{2.3''}$$

where N is the well known asymptotic constant, which in dispersion theory is interpreted as the deuteron-nucleon coupling constant.

The form of the condition (2.3″) strongly suggests the expansion of the wave function as a superposition of decreasing exponentials.

Therefore, we shall write

$$u(r) = \int_0^{\infty} \varrho(\sigma) \exp\left[-\sigma r\right] \mathrm{d}\sigma \,. \tag{2.4}$$

We now substitute equation (2.4) into equation (2.2) and obtain for the spectral function $\varrho(\sigma)$ the following integral equation of the Volterra type

$$(\sigma^2 - \chi^2)\, \varrho(\sigma) = M \int_0^{\sigma - \mu_0} g(\sigma - \sigma')\, \varrho(\sigma')\, \mathrm{d}\sigma' \,. \tag{2.5}$$

By taking into account the asymptotic condition (2.3″) the equation (2.5) can be transformed in the inhomogeneous form

$$\varrho(\sigma) = N\, \delta(\sigma - \chi) + \frac{M}{\sigma^2 - \chi^2} \int_0^{\sigma - \mu_0} g(\sigma - \sigma')\, \varrho(\sigma')\, \mathrm{d}\sigma' \,. \tag{2.6}$$

It is clear that for a given potential $g(\mu)$ equation (2.6) has an unique solution for any value of the parameter χ.

We have, however, to add the boundary condition at the origin (2.3′)

which gives

$$D(\chi) = \int_0^\infty \varrho(\sigma, \chi)\, \mathrm{d}\sigma = 0\,. \tag{2.7}$$

The vanishing of $D(\chi)$ gives the possible eigenvalues of the bound state problem.

The simplest method of solving equation (2.6) is the iteration expansion. We know from the theory of Volterra equations that such an expansion is indeed convergent for any value of the strength of the potential.

The explicit form of the iteration series for $\varrho(\sigma)$ and $D(\chi)$ are

$$\varrho(\sigma) = \varrho_0(\sigma) + \varrho_1(\sigma) + \varrho_2(\sigma) + \ldots + \varrho_n(\sigma) + \ldots\,, \tag{2.8}$$

$$\left\{\begin{aligned}
&\varrho_0(\sigma) = N\,\delta(\sigma-\chi); \qquad \varrho_1(\sigma) = \frac{N\cdot M}{\sigma^2-\chi^2}\, g(\sigma-\chi)\,,\\
&\varrho_2(\sigma) = N M^2 \int_0^{\sigma-\mu_0} \frac{g(\sigma-\sigma_1)}{\sigma^2-\chi^2}\cdot\frac{g(\sigma_1-\chi)}{\sigma_1^2-\chi^2}\,\mathrm{d}\sigma_1,\\
&\varrho_n(\sigma) = N M^n \int_0^{\sigma-\mu_0} \mathrm{d}\sigma_1 \frac{g(\sigma-\sigma_1)}{\sigma^2-\chi^2} \int_0^{\sigma-\mu_0} \mathrm{d}\sigma_2 \frac{g(\sigma_1-\sigma_2)}{\sigma_1^2-\chi^2} \ldots\\
&\qquad\qquad \ldots \int_0^{\sigma_{n-2}-\mu_0} \mathrm{d}\sigma_{n-1} \frac{g(\sigma_{n-2}-\sigma_{n-1})}{\sigma_{n-2}^2-\chi^2}\cdot\frac{g(\sigma_{n-1}-\chi)}{\sigma_{n-1}^2-\chi^2}\,,
\end{aligned}\right. \tag{2.9}$$

$$D(\chi) = N\left(1 + \int_0^\infty \frac{M g(\sigma-\chi)\,\mathrm{d}\sigma}{\sigma^2-\chi^2} + \int_0^\infty \mathrm{d}\sigma \int_0^{\sigma-\mu_0} \mathrm{d}\sigma_1\, M^2\, \frac{g(\sigma-\sigma_1)}{\sigma^2-\chi^2}\cdot\frac{g(\sigma_1-\chi)}{\sigma_1^2-\chi^2} + \ldots\right). \tag{2.10}$$

A very important property of the iteration series (2.8) is that the value of $\varrho(\sigma)$ for $\sigma < \chi + n\mu_0$ is completely determined by the sum of the first n terms of the series.

This property is a consequence of the form of the Volterra equation (2.6) and in particular of the appearence of the upper limit $\sigma - \mu_0$ in the integral. Due to this property the iteration series (2.9) and (2.10) have a rather good convergence. One can verify that for the usual kind of potentials the eigenvalue χ is determined quite accurately by keeping only the first terms of the series. If we consider the wave function (2.4), this property means that the n-th approximation determines the wave function quite accurately in the region

$$r_n > (\chi + n\mu_0)^{-1}\,.$$

We shall see in the next section, that the representation (2.4) for $u(r)$, together with the form of the iteration expansion (2.9), will allow us to localize all the singularities of the vertex functions involving the function $u(r)$.

2·2. – Let us now discuss briefly the generalization of the method to all values of l.

The Schrödinger equation for the wave function $u(r)$ is now

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2}\right) u_l(r) - \chi^2 u_l(r) = M \int_{\mu_0}^{\infty} g^l(\mu) \frac{\exp[-\mu r]}{r} \mathrm{d}\mu\, u_l(r) \,. \tag{2.11}$$

The asymptotic condition for the wave function is

$$u_l(r) \sim N \chi r\, h_l(i\chi r) \,, \qquad r \to \infty \,, \tag{2.12}$$

where $h_l(i\chi r)$ is the Hankel function of order l.

Let us now consider $u_l(r)$ around the origin.

It is well known that the general solution of the equation (2.11) is of the type

$$u_l(r) = a r^{l+1} F_1(r) + b r^{-l} F_2(r) \,, \tag{2.13}$$

where $F_1(r)$ and $F_2(r)$ are two functions regular in the origin.

It is clear that the only physically acceptable solution is for $b = 0$.

This gives the condition

$$\lim_{r \to \infty} r^l u_l(r) = 0 \,. \tag{2.14}$$

The obvious generalization of the Ansatz (2.4) is

$$u_l(r) = -\int_0^{\infty} \varrho_l(\sigma) \sigma r h_l(i\sigma r)\, \mathrm{d}\sigma \,. \tag{2.15}$$

We insert equation (2.15) into (2.11) and use the following identities (see Appendix)

$$\frac{\exp[-\mu r]}{r} h_l(i\beta r) = \int_0^{\infty} G_l(\sigma, \beta, \mu) h_l(i\sigma r)\, \mathrm{d}\sigma \,, \tag{2.16}$$

$$G_l(\sigma, \beta, \mu) = -\frac{\sigma}{\beta} \int_0^{\infty} \mathrm{d}\gamma P_l\left(\frac{\gamma}{\beta}\right) \theta(\gamma - \beta) \frac{\mathrm{d}}{\mathrm{d}\mu} \left[\theta(\sigma - \mu - \gamma) P_l\left(\frac{\mu + \gamma}{\sigma}\right)\right] . \tag{2.17}$$

We thus obtain

$$\varrho_l(\sigma) = N\,\delta(\sigma-\chi) + \frac{M}{\sigma^2-\chi^2}\int_0^{\sigma-\mu_0} K(\sigma,\sigma')\,\varrho_l(\sigma')\,\mathrm{d}\sigma' , \tag{2.18}$$

where

$$K(\sigma,\sigma') = \int_{\mu_0}^{\infty} g'(\mu)\,G_l(\sigma,\sigma',\mu)\,\mathrm{d}\mu . \tag{2.19}$$

It is clear that (2.18) is a Volterra equation like (2.6), which for a given potential $K(\sigma,\sigma')$ has an unique solution for any value of the parameter χ, and has an expansion solution which converges for any value of the strength of the potential.

The bound states are determined by the condition at the origin

$$D_l(\chi) = \int_0^{\infty} \varrho_l(\sigma,\chi)\,\sigma^{-l}\,\mathrm{d}\sigma = 0 . \tag{2.20}$$

It is clear also that if one iterates the series (2.18), the n-th term determines the function $\varrho_l(\sigma)$ up to the point $\chi + n\mu_0$.

Finally, for the further applications, it is useful to write the function $u_l(r)$ as an integral transform of the kind

$$u_l(r) = \int_0^{\infty} G_l(\sigma)\,\exp\,[-\sigma r]\,\mathrm{d}\sigma . \tag{2.21}$$

For this, starting from the integral representation (A.2) for $h_l(i\sigma r)$, one obtains

$$h_l(i\sigma r) = \tag{2.22}$$

$$= \frac{(-i)^{l+2}}{\sigma}\int_0^{\infty}\left[\delta(\gamma-\sigma)\,P_l\left(\frac{\gamma}{\sigma}\right)\frac{\exp\,[-\gamma r]}{r} + \theta(\gamma-\sigma)\,\frac{\exp\,[-\gamma r]}{r}\,\frac{\mathrm{d}}{\mathrm{d}\gamma}\,P_l\left(\frac{\gamma}{\sigma}\right)\right]\mathrm{d}\gamma .$$

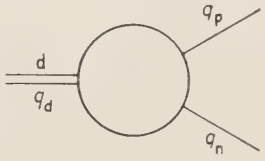
Substituting (2.22) into (2.15) and comparing with (2.21), one finally has

$$G_l(\sigma) = (-i)^l\int_0^{\infty}\varrho_l(\gamma)\left[\delta(\sigma-\gamma)\,P_l\left(\frac{\sigma}{\gamma}\right) + \theta(\sigma-\gamma)\,\frac{\mathrm{d}}{\mathrm{d}\sigma}\,P_l\left(\frac{\sigma}{\gamma}\right)\right]\mathrm{d}\gamma . \tag{2.23}$$

It is clear that the spectrum of $G_l(\sigma)$ coincides with the spectrum of $\varrho_l(\sigma)$.

3. – The vertex functions.

3·1. *The deuteron-proton-neutron vertex.* – We want to discuss the vertex function for $d \to p + n$ and compare the results obtained in the relativistic theory with the ones given by the potential model. As in ref. [5] we shall take the deuteron and the neutron on the mass shell.



$$q_d^2 = -M_d^2,$$
$$q_n^2 = -M^2,$$
$$q_p^2 = t$$

Fig. 1.

and study the vertex $F(t)$ as a function of the square q_p^2 of the proton momentum.

The vertex function $F(t)$ has first of all, a pole for $t = -M^2$, corresponding to a real proton. The residual of the pole is of course, the constant N.

If one considers the higher order in perturbation expansion the analysis of KSW can be applied, giving a cut starting at the so called « anomalous threshold », the values of which in the limit $\chi/M \to 0$, $\mu_0/M \to 0$ is

$$q_p^2 = M^2 + 4\chi\mu_0 + 2\mu_0 .$$

The higher order graphs are rather difficult to investigate, and for the moment no simple rule is available (as in the case of the normal thresholds), in order to locate their singularities.

Let us now turn to our potential approach. The analogous of the vertex function is indeed the Fourier transform of our wave function

$$f(\bar{q}^2) = \int \psi(r) \exp\left[i\bar{q}\,\bar{r}\right] \mathrm{d}_3 r , \tag{3.1}$$

$\bar{q}^2$ is the square of the (three-dimensional) momentum of the proton. Therefore, $\bar{q}^2$ will be connected in the following manner to the relativistic kinematical variables

$$\bar{q}^2 = \frac{[M_d^2 - M^2 - t]^2}{4M_d} + t \simeq -\left(\chi^2 - \frac{t + M^2}{2}\right).$$

(5) Y. NAMBU: *Nuovo Cimento*, **9**, 610 (1958).

By using the expansion (2.4) we obtain

$$f(\bar{q}^2) = 4\pi \int_0^\infty \frac{\varrho(\sigma)\,\mathrm{d}\sigma}{\sigma^2 + \bar{q}^2}\,, \tag{3.2}$$

where $\varrho(\sigma)$ is the weighting function, the explicit form of which is given in equation (2.8).

We thus recognize a pole for $\bar{q}^2 = -\chi^2$ and a series of branch points for $\bar{q}^2 = -(\chi+\mu_0)^2$; $\bar{q}^2 = -(\chi+2\mu_0)^2$; ... .

The pole and the first branch point are indeed identical to the ones obtained by considering the relativistic graphs (fig. 2-*a*), (fig. 2-*b*).

The further branches $-(\chi+n\mu_0)^2$ correspond to the higher order graphs like (fig. 2-*c*), corresponding to the exchange of many pions between the nucleons.

It is amusing to remark that also the location of such singularities is given by a very simple formula. It would be very desirable to have a method which gave directly such a result in the framework of a relativistic theory.

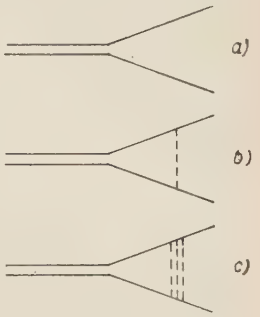


Fig. 2.

3·2. *The electromagnetic form factor.* – In strict analogy with the $\mathrm{d} \to \mathrm{n} + \mathrm{p}$ vertex function, we shall now discuss the electromagnetic form factor of the deuteron, comparing the results of the relativistic perturbation theory with the ones obtained in the potential model.

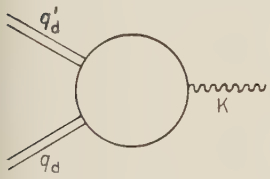


Fig. 3.

The electromagnetic vertex corresponds to the diagram of Fig. 3, where the deuteron is on the mass shell $q_d^2 = q_d'^2 = -M_d^2$ and we shall study the vertex function $F(k^2)$ as a function of the square k^2 of the photon momentum.

The lowest order diagram in perturbation theory is given in Fig. 4. Using the method of KSW one can write such a term in the spectral form,

$$F(k^2) = \int_{\lambda_0^2}^\infty \frac{G(\lambda^2)\,\mathrm{d}\lambda^2}{k^2 + \lambda^2}\,, \tag{3.3}$$

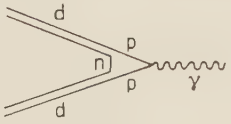


Fig. 4.

where the lowest limit of integration is, in the limit $B/M \to 0$,

$$\lambda_0^2 = 16\,MB\,. \tag{3.4}$$

Let us now compare with the potential model. We have (6)

$$F(k^2) = \int \psi^*(\bar{r}) \exp\left[i\bar{k}\bar{r}/2\right] \psi(\bar{r})\, \mathrm{d}_3 r = 4\pi \int_0^\infty u^2(r)\, j_0(kr/2)\, \mathrm{d}r \,. \tag{3.5}$$

Using equation (2.4) we can write

$$F(k^2) = \int_0^\infty \frac{G(\lambda^2)\, \mathrm{d}\lambda^2}{k^2 + \lambda^2} \,, \tag{3.6}$$

and we get for the spectral $G(\lambda^2)$

$$G(\lambda^2) = \frac{4\pi}{\lambda} \int_0^\infty \mathrm{d}\sigma \int_0^\infty \mathrm{d}\sigma'\, \varrho(\sigma)\, \varrho(\sigma')\, \theta\big(\lambda - 2(\sigma + \sigma')\big) \,. \tag{3.7}$$

Looking at the expansion (2.8) of the weight function $\varrho(\sigma)$ one can immediately recognize the location of the singularities of $F(k^2)$. We have a series of branch points at $k^2 = -(4\chi)^2$; $-(4\chi + 2\mu_0)^2$; ..., $-(4\chi + 2n\mu_0)^2$; ... . Also here the first branch point of the potential theory coincides with the one given by the first order relativistic diagram (Fig. 4) (7).

3·3. *Higher angular momenta.* – We want to discuss now the case in which the bound state has a higher angular momentum.

For the vertex $\mathrm{d} \to \mathrm{p} + \mathrm{n}$ function we can write

$$\int \psi_{lm}(\bar{x}) \exp\left[i\bar{q}\,\bar{x}\right] \mathrm{d}_3 x = f_l(q^2)\, Y_{lm}(\bar{u}) \,, \tag{3.8}$$

where

$$\bar{u} = \frac{\bar{q}}{q} \,; \qquad f_l(q^2) = \int_0^\infty u_l(r)\, j_l(qr)\, \mathrm{d}r \,. \tag{3.9}$$

Using the expression (2.15) for $u_l(r)$ we obtain

$$f_l(q^2) = 4\pi q^l \int_0^\infty \frac{\varrho_l(\sigma)\, \sigma^{-l}}{\sigma^2 + q^2}\, \mathrm{d}\sigma \,, \tag{3.10}$$

---

(6) The appearance in the exponential of the factor $kr/2$ is due to the fact that $\boldsymbol{r}$ is the proton-neutron distance, and $r/2$ is the distance between the proton and the center of mass of the composite system.

(7) R. OEHME: *Nuovo Cimento*, **13**, 778 (1959).

we see that the location of the singularities is the same as in the case of S-wave.

The electromagnetic vertex is given by the integral

$$\int \psi^*_{lm}(\bar{x}) \exp\left[\frac{i\bar{k}\bar{x}}{2}\right] \psi_{lm}(\bar{x})\, \mathrm{d}_3\bar{x} = V^l_{mm'}(\bar{k})\,. \tag{3.11}$$

Such an integral can be expressed in terms of the $2l$ scalar functions $f_i(k^2)$

$$V^l_{mm'}(\bar{k}) = \sum_{i=0}^{2l} f_i(k^2) I_i^{mm'}(\bar{u})\,, \tag{3.12}$$

where

$$I_i^{mm'}(\bar{u}) = \sum_{\mu} C^{mm'\mu}_{l\,l\,i}\, Y^{\mu}_i(\bar{u})\,, \tag{3.13}$$

$C^{mm'\mu}_{l\,l\,i}$ are Clebsch-Gordon coefficients and $\bar{u} = \bar{k}/k$. The functions $f_i(k^2)$ are given by

$$f_i(k^2) = \int u_l^2(r) j_i\left(\frac{kr}{2}\right) \mathrm{d}r\,. \tag{3.14}$$

Using now the expression (2.21) for $u_l(r)$, we obtain, in close analogy with the s-wave case, the following representation for

$$f_i(k^2) = \int_{\lambda_0^2}^{\infty} \frac{G_i(\lambda^2)\, \mathrm{d}\lambda^2}{k^2 + \lambda^2}, \tag{3.15}$$

where

$$G_i(\lambda^2) = \frac{k^i}{(\lambda)^{i+1}} \int_0^{\infty} \mathrm{d}\sigma \int_0^{\infty} \mathrm{d}\tau\, \theta(\lambda - 2(\sigma + \tau))\, G_l(\sigma) G_l(\tau)\, 4P_l\left(\frac{2(\sigma + \tau)}{\lambda}\right). \tag{3.16}$$

Those results are of course expected, because we know from field theory that the form of the spectral representation does not depend on the spin of the particle involved.

It is interesting to note that although the integrals in equation (3.17) are convergent, because $u_l(r)$ is well behaved at the origin, if one substitutes the perturbative expansion of $G_l(\sigma)$ into equation (3.15), (3.16), the integrals at each order does diverge for large σ.

This is due to the fact that a well behaved function at the origin is written as superposition of singular Hankel functions.

The regularity condition (2.20) would ensure that the total result of equation (3.15) is convergent.

* * *

We wish to thank Prof. S. FUBINI for having suggested this analysis and his continuous assistance throughout the work, and Prof. R. STROFFOLINI for useful discussions.

One of us (L.B.) wishes also to thank his Colonel, U. L. RONCA, who with great interest in science has permitted his collaboration to this work.

APPENDIX

To demonstrate eq. (2.17) we shall use the following identities

$$j_l(kr) = \frac{1}{2i^l}\int_{-1}^{+1} P_l(x)\exp[ikrx]\,\mathrm{d}x\,, \tag{A.1}$$

$$h_l(i\sigma r) = \frac{(-i)^{l+2}}{\sigma}\int_{\sigma}^{\infty}\mathrm{d}\gamma\, P_l\left(\frac{\gamma}{\sigma}\right)\exp[-\gamma r]\,, \tag{A.2}$$

$$\int_0^{\infty}\exp[-ar]\,j_l(kr)\,\mathrm{d}r = \int_a^{\infty}\left(\frac{k}{\beta}\right)^l \frac{P_l(a/\beta)\,\mathrm{d}\beta}{k^2+\beta^2}\,. \tag{A.3}$$

For (A.1) and (A.2) see [8]. We shall derive eq. (A.3). Using eq. (A.1) one has

$$\int_0^{\infty}\exp[-ar]\,j_l(kr)\,\mathrm{d}r = \tag{A.4}$$

$$= \frac{1}{2i^l}\int_0^{\infty}\mathrm{d}r\int_{-1}^{+1}\mathrm{d}x\,P_l(x)\exp[-(a-ikx)r] = \frac{1}{2i^l}\int_{-1}^{+1}\frac{P_l(x)\,\mathrm{d}x}{a-ikx}\,.$$

([8]) P. M. MORSE and N. FESHBACH: *Methods of Theoretical Physics*, p. 1467.

Using now the identity

$$\text{(A.5)} \qquad \frac{1}{A+B} = \frac{1}{A}\left[1 + \left(-\frac{B}{A}\right) + \left(-\frac{B}{A}\right)^2 + \ldots + \left(-\frac{B}{A}\right)^{l-1}\right] + \\ + (-1)^l \left(\frac{B}{A}\right)^l \frac{1}{A+B},$$

and the orthogonality of the Legendre polynomials (the first term on the right hand side of (A.5) can be expressed by means of a sum of Legendre polynomials of order less than l) one can write

$$\text{(A.6)} \qquad \int_0^\infty \exp[-ar] j_l(kr)\,\mathrm{d}r = \frac{1}{2i^l}\int_{-1}^{+1} \frac{P_l(x)(-1)^l(-ikx)^l\,\mathrm{d}x}{(a-ikx)a^l} = \frac{1}{2}\int_{-1}^{+1} \frac{P_l(x)k^l x^l \mathrm{d}x}{(a-ikx)a^l},$$

and with the substitution $x = 1/u$

$$\text{(A.7)} \qquad \int_0^\infty \exp[-ar] j_l(kr)\,\mathrm{d}r = -\frac{1}{2}\left[\int_{-1}^{-\infty} \frac{P_l(1/u)}{a-ik/u}\frac{k^l}{a^l u^l}\frac{\mathrm{d}u}{u^2} + \int_{\infty}^{1} \frac{P_l(1/u)}{a-ik/u}\frac{k^l}{a^l u^l}\frac{\mathrm{d}u}{u^2}\right] = \\ = \int_1^\infty \frac{P_l(1/u)au}{(au)^2+k^2}\frac{k^l}{a^l u^l}\frac{\mathrm{d}u}{u} = \int_\beta^\infty \frac{P_l(a/\beta)k^l \mathrm{d}\beta}{\beta^l(\beta^2+k^2)}.$$

We can now easily derive eq. (2.17). Multiplying eq. (2.16) by $j_l(kr)r^2$ and integrating

$$\text{(A.8)} \qquad \int_0^\infty j_l(kr) r^2 \frac{\exp[-\mu r]}{r} \cdot h_l(i\beta r) = \int_0^\infty \mathrm{d}\sigma G(\sigma, \beta, \mu) \int_0^\infty \mathrm{d}r\, r^2 j_l(kr) h_l(i\sigma r).$$

Using now the relation

$$\int_0^\infty r^2 j_l(kr) h_l(i\sigma r)\,\mathrm{d}r = -\frac{(-ik)^l}{\sigma^{l+1}}\frac{1}{\sigma^2+k^2},$$

one has

$$\text{(A.9)} \qquad \int_0^\infty j_l(kr) r \exp[-\mu r] h_l(i\beta r)\,\mathrm{d}r = -\int_0^\infty G(\sigma, \beta, \mu)\frac{(-ik)^l}{\sigma^{l+1}}\frac{\mathrm{d}\sigma}{\sigma^2+k^2}.$$

The left hand side of eq. (A.9) can be written, using eq. (A.2) and (A.7)

$$\text{(A.10)}\qquad \int_0^\infty j_l(kr) r \exp[-\mu r] h_l(i\beta r)\,\mathrm{d}r = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\mu}\left[j_l(kr)\exp[-\mu r]h_l(i\beta r)\right]\mathrm{d}r =$$

$$= \frac{(-i)^l}{\beta}\int_\beta^\infty \mathrm{d}\gamma P_l\left(\frac{\gamma}{\beta}\right)\frac{\mathrm{d}}{\mathrm{d}\mu}\int_0^\infty \mathrm{d}r \exp[-(\mu+\gamma)r] j_l(kr) =$$

$$= \frac{(-i)^l}{\beta}\int_\beta^\infty \mathrm{d}\gamma P_l\left(\frac{\gamma}{\beta}\right)\frac{\mathrm{d}}{\mathrm{d}\mu}\int_{\mu+\gamma}^\infty \frac{k^l}{\sigma^l}\frac{P_l((\mu+\gamma)/\sigma)\,\mathrm{d}\sigma}{k^2+\sigma^2} =$$

$$= \int_0^\infty \frac{(-ik)^l}{\sigma^l}\frac{1}{\sigma^2+k^2}\frac{1}{\beta}\int_0^\infty \mathrm{d}\gamma\,\theta(\gamma-\beta)P_l\left(\frac{\gamma}{\beta}\right)\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\theta(\sigma-\mu-\gamma)P_l\left(\frac{\mu+\gamma}{\sigma}\right)\right].$$

Comparing (A.9) and (A.10) one obtains finally eq. (2.17)

$$G(\sigma,\beta,\mu) = -\frac{\sigma}{\beta}\int_0^\infty \mathrm{d}\gamma\,\theta(\gamma-\beta)P_l\left(\frac{\gamma}{\beta}\right)\frac{\mathrm{d}}{\mathrm{d}\mu}\left|\theta(\sigma-\mu-\gamma)P_l\left(\frac{\mu+\gamma}{\sigma}\right)\right|.$$

RIASSUNTO

Facendo uso di un modello di potenziale sono state studiate le proprietà analitiche delle funzioni di vertice di una particella composta (« deutone »). Le singolarità di queste funzioni sono state confrontate con le singolarità che si ottengono facendo uso della teoria perturbativa relativistica.

IL NUOVO CIMENTO Vol. XVIII, N. 4 16 Novembre 1960

Potential Model of Deuteron Photodisintegration and Mandelstam Representation.

V. De Alfaro and C. Rossetti

Istituto di Fisica dell'Università - Torino
Istituto Nazionale di Fisica Nucleare - Sezione di Torino

(ricevuto il 28 Luglio 1960)

Summary. — In this paper the analytic properties of the fixed angular momentum amplitudes of the deuteron photodisintegration are investigated by a potential model. It is shown that these properties agree with those obtained by a Mandelstam representation with anomalous thresholds, as argued by perturbation arguments. This is an argument in favour of the validity of general dispersion techniques also in nuclear physics.

1. – Introduction.

In the recent times a good deal of attention has been given to potential scattering in the first quantization theory. The interest arises from the cleverness in showing, by means of potential methods, the analyticity properties of the scattering matrix elements which had only been guessed in a more general relativistic theory. Indeed, the double dimensional Mandelstam representation, which is at present only supported by perturbation calculations in field theory, has been obtained [1] in non relativistic potential methods starting from a Schrödinger equation. Moreover it has been shown that in some cases, as in nucleon-nucleon scattering, at least until anelastic phenomena do not play an important role, the potential framework is a suitable description of $\mathcal{N}$-$\mathcal{N}$ scattering phenomena [2]. The interest in potential problems is therefore due to the better understanding one gets in this way about physical phenomena, in spite of the limited validity of the proved results.

(1) See for reference: A. Klein: *Journ. Math. Phys.*, **1**, 41 (1960).
(2) J. M. Charap and S. P. Fubini: *Nuovo Cimento*, **14**, 540 (1959) and **15**, 73 (1960).

A very interesting and still open question is the investigation of the analytic properties of amplitudes in processes involving composite particles. The problem is to see if dispersion techniques, as double integral representations, are valid in non « elementary » particles physics, and therefore if they can be used as a useful method also in nuclear physics.

A considerable effort has been done in treating formally on the same footing elementary and so-called composite particles. Indeed it has been shown that for composite particles it is possible to define in- and out-fields, as for the elementary ones; so that one can hope that the analytic properties of the amplitudes in processes involving bound states could be obtained by applying general principles, but such a program is very complicated.

On the other hand, perturbation theory in the lowest orders has been used to obtain the analytic properties. Such a method has the advantage to be a covariant one, but it has no hamiltonian grounds on which to draw the Feynman graphs; furthermore it is almost impossible to get the analytic properties of the higher order graphs.

On the grounds of such perturbation studies, KARPLUS, SOMMERFIELD and WICHMANN ([3]), have shown that if composite particles are considered in a process, then the matrix elements admit spectral representations showing anomalous thresholds. They are « anomalous » in the sense that the imaginary part of the matrix element is different from zero well before intermediate states are possible which conserve the energy. The appearance of such anomalous analytic behaviour is directly related to the dynamical structure of the considered particles, *i.e.* they come out from the « composite » structure of the particle, which is connected to the smallness of the binding energy of composing particles ([4]).

In trying to extend the dispersion techniques to nuclear physics, the study of the analytic properties by potential models would give valuable information about the possibility of using spectral representations and so on. It is clear that whenever one goes on to the non relativistic limit of dispersion relations or Mandelstam representations, one should obtain the potential model results. The study by potential models clarifies the origin of the anomalous thresholds, and besides can give an insight towards the use of Feynman graphs in relativistic perturbation theory.

BLANKENBECLER and COOK ([5]) have shown that a dispersion treatment of the d-$\mathcal{N}\mathcal{N}$ vertex gives the same results as Schrödinger equation, which allows to define a nucleon-nucleon potential describing the bound state properties

---

([3]) R. KARPLUS, C. M. SOMMERFIELD and E. M. WICHMANN: *Phys. Rev.*, **111**, 1187 (1958) and **114**, 376 (1959).

([4]) R. OEHME: *Nuovo Cimento*, **13**, 778 (1959).

([5]) R. BLANKENBECLER and L. F. COOK jr.: preprint.

when used in conjunction to a Schrödinger equation. In the same spirit the first problems one may consider are, as examples of scattering and vertex type problems respectively, the photodisintegration of the deuteron and the deuteron form factor. We shall investigate the photodisintegration problem. The form factor has been treated in a parallel way by BERTOCCHI, CEOLIN, and TONIN (6).

The final task in the photodisintegration problem should be to obtain a Mandelstam representation in potential theory. Indeed it is known that for the lowest orders in perturbation theory the double integral representation is valid also when anomalous thresholds appear (7). In this paper we limit ourselves to investigate, by means of a potential model, the analytic properties of the fixed angular momentum matrix elements. We take the nucleon-nucleon potential as a superposition of Yukawa potentials, and the potential is chosen so to reproduce the bound state and scattering properties if inserted into a Schrödinger equation.

We start by showing the analytic behaviour of the $l = 0$ matrix element; then we extend the method to any angular momentum. In doing this we use a representation of wave functions which is due to A. MARTIN (8), and which has also been used in momentum space by BLANKENBECLER and COOK. Finally a Mandelstam representation, with the thresholds given by the lowest order perturbation theory, is written for the photodisintegration amplitude, and it is easily seen that the analytic properties for fixed l given by this representation agree with the potential model results. We treat of course the ideal problem of scalar particles. The analyticity properties however depend only on the rough features of the intermediate mass spectrum, and on the initial and final masses, and not on such complications as spins or gauge and so on.

2. – Matrix element of deuteron photodisintegration.

We wish to discuss the photodisintegration of a bound state of mass M in two scalar particles of equal masses m. The binding energy is $2m - M = b = \alpha^2/m \ll m$. For sake of definiteness we will refer to them as deuteron, proton and neutron. It is well known that the photodisintegration amplitude in first quantization is written as follows (9)

$$F(q, \cos\theta) = (\psi_d(r), \exp[i\,\boldsymbol{k}\cdot\boldsymbol{x}_p]\Psi) ,$$

(6) L. BERTOCCHI, C. CEOLIN and M. TONIN: *Nuovo Cimento*, to be published.
(7) S. MANDELSTAM: *Phys. Rev.*, **115**, 1741 (1959).
(8) A. MARTIN: *Nuovo Cimento*, **14**, 403 (1959) and **15**, 99 (1960).
(9) We use natural units: $\hbar = c = 1$.

where $\boldsymbol{x}_p$ is the proton co-ordinate, which is related to $\boldsymbol{X}$, the co-ordinate of the center of mass between proton and neutron, and to the relative co-ordinate, $\boldsymbol{r}$, by $\boldsymbol{x}_p = \boldsymbol{X} + (\boldsymbol{r}/2)$.

$\psi_d(r)$ is the wave function describing the bound state, which we suppose to be a pure S state. Ψ describes the final state of the two outgoing nucleons:

$$\Psi = \boldsymbol{\psi}_{out}(\boldsymbol{q}, \boldsymbol{r}) \exp [i\boldsymbol{K}\cdot\boldsymbol{X}] .$$

$\psi_{out}(\boldsymbol{q}, \boldsymbol{r})$ is the solution of the Schrödinger equation corresponding to nucleon-nucleon scattering. $\boldsymbol{q}$ is the final relative momentum in the center of mass system, $\boldsymbol{k}$ is the photon momentum, which is related to $\boldsymbol{q}$ by the energy conservation, and $\boldsymbol{q}\boldsymbol{k} = qk\cos\vartheta$.

By separating the center of mass motion, the matrix element, apart from a δ-function over the momenta, becomes

$$F(q, \cos\theta) = (\psi_d(\boldsymbol{r}), \exp [i\boldsymbol{k}\cdot\boldsymbol{r}/2]\, \psi_{out}(\boldsymbol{q}, \boldsymbol{r})) ,$$

which can be written

$$F(q, \cos\theta) = \sum_0^\infty{}_l A_l(q) P_l(\cos\theta) , \tag{1}$$

where $A_l(q)$ is the photodisintegration amplitude corresponding to the l-th angular momentum wave:

$$A_l(q) = \int_0^\infty \mathrm{d}r\, r^2 \psi_d(r)\, j_l(kr/2)\, \psi_l(q, r) . \tag{2}$$

$\psi_l(q, r)$ is the solution of the radial Schrödinger equation which behaves at infinity as

$$\frac{\exp [i\delta_l]}{qr} \cos\left\{qr + \delta_l - (l+1)\frac{\pi}{2}\right\} .$$

3. – S-wave matrix element.

3·1. – The $l = 0$ matrix element can be written as follows

$$A_0(q) = \frac{2}{k}\int_0^\infty \mathrm{d}r\, \varphi_d(r) \frac{\sin (kr/2)}{r} \varphi(q, r) . \tag{3}$$

$\varphi_d(r)$ is the solution of the radial Schrödinger equation

$$\varphi_d''(r) - \alpha^2 \varphi_d(r) = V(r)\varphi_d(r)\,, \tag{4}$$

which describes the deuteron.

$\varphi(q, r)$ satisfies the scattering equation

$$\varphi''(q, r) + q^2 \varphi(q, r) = V(r)\varphi(q, r) \tag{5}$$

and behaves at infinity as $\exp[i\delta]\sin(qr+\delta)$.

We suppose, according to ref. ([8]), that the potential in (4) and (5) can be represented as an integral of the following type:

$$V(r) = f\int_0^\infty g(\sigma)\exp[-\sigma r]\,\mathrm{d}\sigma\,, \tag{6}$$

with the condition $g(\sigma) = 0$ if $\sigma < \mu$ (μ is the π-meson mass), which is a very plausible representation for the physical potential.

Now, if we put

$$\varphi_d(r) = \int_0^\infty \eta(\sigma)\exp[-\sigma r]\,\mathrm{d}\sigma\,, \qquad \eta(\sigma) = 0 \text{ if } \sigma < 0\,, \tag{7}$$

replacing $\varphi_d(r)$ with this expansion into formula (4), one obtains the following equation for the spectral function η:

$$\eta(\sigma) = \delta(\sigma - \alpha) + f\,\frac{1}{\sigma^2 - \alpha^2}\int_0^{\sigma-\mu}\mathrm{d}\beta\,\eta(\beta)g(\sigma-\beta)\,, \tag{8}$$

where the deuteron wave function has been normalized so that its asymptotic value for large r is $\exp[-\alpha r]$.

Also $\varphi(q, r)$ can be expanded in a very similar way; namely we can write it in the form

$$\varphi(q, r) = A(-q, r)\exp[-iqr] - S(q)A(q, r)\exp[iqr]\,, \tag{9}$$

where $A(q, r) \to 1$ when $r \to \infty$.

From the regularity of the wave function in $r = 0$ follows the definition of

the S-matrix:

$$S(q) = \frac{A(-q)}{A(q)} \qquad \text{with} \qquad A(q) = A(q, 0).$$

By inserting (9) into eq. (5) one obtains for $A(q, r)$

$$A''(q, r) - 2iqA'(q, r) = V(r)A(q, r). \tag{10}$$

Let us now expand (just in the same manner as for the bound state wave function) the function $A(q, r)$ by means of Laplace integrals,

$$A(q, r) = \int_0^\infty d\sigma \lambda(q, \sigma) \exp[-\sigma r], \qquad \lambda(q, \sigma) = 0 \text{ if } \sigma < 0.$$

Putting this « Ansatz » into eq. (10) we obtain

$$\lambda(q, \sigma) = \delta(\sigma) + f \frac{1}{\sigma(\sigma - 2iq)} \int_0^{\sigma-\mu} d\beta \lambda(q, \beta) g(\sigma - \beta), \tag{11}$$

where the asymptotic behaviour of $A(q, r)$ at infinity has been taken into account in writing the δ-function in (11).

It is very interesting to notice some peculiar features of the spectral functions η and λ, as they appear from eqs. (8) and (11). The support of η is the point $\sigma = \alpha$ and the continuum $\sigma \geqslant \alpha + \mu$; the support of λ is $\sigma = 0$ and the continuum $\sigma \geqslant \mu$. The n-th order in the coupling constant f determines exactly the spectral function up to a value of σ equal to $n\mu$ over the continuum threshold. One sees in this way that increasing orders in f correspond to determine $\varphi_d(r)$ or $A(q, r)$ in smaller and smaller regions of r, according to the physical intuition that higher order corrections give the right form of the wave function in inner regions.

3·2. – By inserting the preceding results for the wave functions in the expression (3), one obtains

$$\begin{aligned} A_0(q) = I_\alpha(k, -q) - S(q) I_\alpha(k, q) + \int_{\alpha+\mu}^\infty d\sigma \eta(\sigma) \{I_\sigma(k, -q) - S(q) I_\sigma(k, q)\} + \\ + \int_\mu^\infty d\sigma \{\lambda(-q, \sigma) I_{\alpha+\sigma}(k, -q) - S(q) \lambda(q, \sigma) I_{\alpha+\sigma}(k, q)\} + \\ + \int_{\alpha+\mu}^\infty d\sigma \eta(\sigma) \int_{\sigma+\mu}^\infty d\gamma \{\lambda(-q, \gamma - \sigma) I_\gamma(k, -q) - S(q) \lambda(q, \gamma - \sigma) I_\gamma(k, q)\}, \end{aligned} \tag{12}$$

where we have put

$$\frac{2}{k}\int_0^\infty \frac{\mathrm{d}r}{r} \exp\left[(iq-\alpha)r\right] \sin\frac{k}{2}r = I_\alpha(k, q)\,. \tag{13}$$

By inspection of expression (12) we find two types of singularities in the matrix element A_0: some which arise from the $S(q)$ function, and some others which depend on the type of the integrals (13).

The singularities of the S-matrix in the first Riemann sheet of the energy (q^2) plane are well known ([10]); $S(q)$ has a pole which is due to a zero of $A(q)$ in $q^2 = -\alpha^2$ (it represents the only bound state). There is further an obvious branch point in $q^2 = 0$, and a second branch point in $q^2 = -\mu^2/4$, which is a consequence of being the potential a continuous superposition of exponential functions. This last singularity, however, disappears in the amplitude (12). Indeed $A_0(q)$ contains $\varphi(q, r)$ which is the ratio $-2iq\,\varphi_{\mathrm{reg}}(q, r)/A(-q)$, where $\varphi_{\mathrm{reg}}(q, r)$ is the so-called regular solution of (5), *i.e.* $\varphi_{\mathrm{reg}}(q, 0) = 0$, $\varphi'_{\mathrm{reg}}(q, 0) = 1$. Now $\varphi_{\mathrm{reg}}(q, r)$ is an entire function of q for potentials of the type (6), while $A(-q)$ is a well behaved function in the $\mathrm{Im}\, q > 0$ half plane (it has zeros on the imaginary axis, corresponding to bound states); hence the branch point in $q^2 = -\mu^2/4$ is not present in our matrix element.

It might be amusing to see how the cut starting from $q = i\mu/2$ disappears in first perturbation order approximation of (12); we do it in Appendix C. The singularity in $q = -i\mu/2$ is present but it is in the second sheet of the energy plane.

We have now to take into account in (12) the singularities deriving from the integrals I. They can be easily seen by writing $I_\alpha(k, q)$ in the form

$$I_\alpha(k, q) = \frac{1}{ik}\log\frac{\alpha - i(q - k/2)}{\alpha - i(q + k/2)}\,,$$

which shows that $I_\alpha(k, q)$ has branch points in $(q + k/2)^2 = -\alpha^2$ and $(q - k/2)^2 = -\alpha^2$.

By inspection of formula (12), one sees that $A_0(q)$ has branch points in $(q \pm k/2)^2 = -\alpha^2$ and branch lines for $(q + k/2)^2 \leqslant -(\alpha+\mu)^2$ and $(q - k/2)^2 \leqslant -(\alpha+\mu)^2$.

4. – Extension to any angular momentum.

4·1. – In the case of any angular momentum we shall limit ourselves to consider a pure Yukawa potential, just in order to avoid complications. The

([10]) See, for example, ref. ([8]).

final nucleon-nucleon wave function is the solution of the Schrödinger equation

$$(D_l + q^2)\psi_l(q, r) = f\frac{\exp[-\mu r]}{r}\psi_l(q, r), \tag{14}$$

where

$$D_l = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{l(l+1)}{r^2}.$$

The asymptotic behaviour of $\psi_l(q, r)$ is

$$\psi_l(q, r) \underset{r\to\infty}{\sim} h_l(-q, r) + (-)^l S_l(q) h_l(q, r) \tag{15}$$

(the h's are spherical Hankel functions of the first kind). Furthermore $\psi_l(q, r)$ is required to be regular in $r = 0$. It is known that the only possible solutions of (14) behave in zero as r^{-l-1} or as r^l; thus, because of the regularity, we require that

$$\lim_{r\to 0} r^{l+1}\psi_l(q, r) = 0. \tag{16}$$

We put now

$$\psi_l(q, r) = R_l(-q, r) + (-)^l S_l(q) R_l(q, r), \tag{17}$$

where R is a solution of (14) which behaves at infinity as

$$R_l(q, r) \underset{r\to\infty}{\sim} h_l(q, r). \tag{18}$$

Just in the same way as for the S-wave, we shall make for R the following « Ansatz », which reveals to be very useful:

$$R_l(q, r) = \int_0^\infty \mathrm{d}\sigma \left(\frac{q + i\sigma}{q}\right)^{l+1} \varrho_l(q, \sigma) h_l[(q + i\sigma)r]. \tag{19}$$

Taking into account eq. (14) for R, we obtain the equation for ϱ_l

$$\int_0^\infty \mathrm{d}\sigma\, \varrho_l(q, \sigma)(q + i\sigma)^{l+1}\sigma(\sigma - 2iq) h_l[(q + i\sigma)r] = \tag{20}$$

$$= f\frac{\exp[-\mu r]}{r}\int_0^\infty \mathrm{d}\sigma\, \varrho_l(q, \sigma)(q + i\sigma)^{l+1} h_l[(q + i\sigma)r].$$

In order to obtain an integral equation for $\varrho_l(q, \sigma)$, we introduce the following identity:

$$(21) \qquad \frac{\exp[-\mu r]}{r}(q+i\sigma)^{l+1} h_l[(q+i\sigma)r] = \int_0^\infty (q+i\beta)^{l+1} h_l[(q+i\beta)r] H_\mu^l(q; \beta, \sigma)\, d\beta .$$

The function $H_\mu^l(q; \beta, \sigma)$ has been evaluated in Appendix A, where it is shown to be

$$(22) \qquad H_\mu^l(q; \beta, \sigma) = \left(\frac{q+i\sigma}{q+i\beta}\right)^l \theta(\beta-\mu-\sigma)\left\{P_l\left(\frac{q+i(\beta-\mu)}{q+i\sigma}\right) - \right.$$
$$\left. - \frac{i}{q+i\beta}\int_\sigma^{\beta-\mu} d\gamma\, P_l\left(\frac{q+i\gamma}{q+i\sigma}\right) P_l'\left(\frac{q+i(\gamma+\mu)}{q+i\beta}\right)\right\} .$$

By use of (21) in eq. (20), and interchanging the order of integration (after a reappealing of the variables β and σ), we get for $\varrho_l(q, \sigma)$ the following integral equation:

$$(23) \qquad \sigma(\sigma - 2iq)\varrho_l(q, \sigma) = f\int_0^{\sigma-\mu} d\beta\, \varrho_l(q, \beta) H_\mu^l(q; \sigma, \beta) ,$$

where the θ-function appearing in (22) has been taken into account in writing the limits of integration. Reminding the asymptotic behaviour of $R_l(q, r)$, given by (18), the integral equation (23) can be written as follows:

$$(24) \qquad \varrho_l(q, \sigma) = \delta(\sigma) + f\frac{1}{\sigma(\sigma-2iq)}\int_0^{\sigma-\mu} d\beta\, \varrho_l(q, \beta) H_\mu^l(q; \sigma, \beta) .$$

This equation is very similar in the form to the S-wave one. It is seen that, because of the peculiar form of the kernel (22), it can be solved by iteration, and that the n-th perturbative order in f is rigorous if $\sigma < n\mu$. This is a consequence of the representation of $R_l(q, r)$ by means of an integral over Hankel functions, so that the effect of the centrifugal potential is contained in these functions.

From the preceding discussion it follows that (19) can be written as follows:

$$(25) \qquad R_l(q, r) = h_l(qr) + \int_\mu^\infty d\sigma\, \varrho_l(q, \sigma)\left(\frac{q+i\sigma}{q}\right)^{l+1} h_l[(q+i\sigma)r] .$$

By inserting the expression (17) for $\psi_l(q, r)$ into the regularity condition one has

$$\lim_{r\to 0} r^{l+1}\{R_l(-q, r) + (-)^l S_l(q) R_l(q, r)\} = 0 ,$$

which allows to define the S-matrix

$$S_l(q) = (-)^{l+1} \frac{\lim_{r\to 0} r^{l+1} R^l(-q, r)}{\lim_{r\to 0} r^{l+1} R_l(q, r)} .$$

In analogy to the S-wave case, one can write the S-matrix as

$$S_l(q) = (-)^{l+1} \frac{R_l(-q)}{R_l(q)} , \tag{26}$$

where

$$R_l(q) = \lim_{r\to 0} r^{l+1} R_l(q, r) . \tag{27}$$

Using representation (25), (27) becomes

$$R_l(q) = \lim_{r\to 0} r^{l+1} \left\{ h_l(q, r) + \int_\mu^\infty d\sigma\, \varrho_l(q, \sigma) \left(\frac{q + i\sigma}{q}\right)^{l+1} h_l[(q + i\sigma)r] \right\} . \tag{28}$$

If now we replace the spherical Hankel function with its behaviour in the neighbourhood of the origin (the validity of such an assumption will be justified in Appendix B) we get for $R_l(q)$ the expression

$$R_l(q) = -\frac{(2l-1)!!}{q^{l+1}} \left\{ 1 + \int_\mu^\infty \varrho_l(q, \sigma)\, d\sigma \right\} . \tag{29}$$

Then the S-matrix (25) can be written

$$S_l(q) = \frac{1 + \int_\mu^\infty d\sigma\, \varrho_l(-q, \sigma)}{1 + \int_\mu^\infty d\sigma\, \varrho_l(q, \sigma)} . \tag{30}$$

Finally, taking into account (26), the scattering wave function $\psi_l(q, r)$ is written in the usual way

$$\psi_l(q, r) = R_l(-q, r) - \frac{R_l(-q)}{R_l(q)} R_l(q, r) . \tag{31}$$

4·2. – Our task is now to investigate the analyticity properties of the photodisintegration matrix elements $A_l(q)$, defined in eq. (2). For sake of simplicity we shall retain for the deuteron wave function only its asymptotic form $\exp[-\alpha r]/r$. The correct form would give complications, because of the further integration on the deuteron spectral function, but it is obvious that the results on the analyticity we shall find are exactly the same.

With the aid of formula (17) we obtain

$$A_l(q) = \int_0^\infty \mathrm{d}r\, r \exp[-\alpha r]\, j_l(kr/2) \left\{ R_l(-q, r) - \frac{R_l(-q)}{R_l(q)} R_l(q, r) \right\}. \tag{32}$$

By inserting the representation (25), the photodisintegration amplitude becomes

$$A_l(q) = \mathscr{I}^l_\alpha\left(\frac{k}{2}, -q\right) - \frac{R_l(-q)}{R_l(q)} \mathscr{I}^l_\alpha\left(\frac{k}{2}, q\right) + \tag{33}$$

$$+ \int_\mu^\infty \mathrm{d}\sigma \left\{ \varrho_l(-q, \sigma) \mathscr{I}^l_\lambda\left(\frac{k}{2}, -q + i\sigma\right) - \frac{R_l(-q)}{R_l(q)} \varrho_l(q, \sigma) \mathscr{I}^l_\lambda\left(\frac{k}{2}, q + i\sigma\right) \right\},$$

where

$$\mathscr{I}^l_\alpha\left(\frac{k}{2}, q + i\sigma\right) = \int_0^\infty \mathrm{d}r\, r\, j_l(kr/2) \left(\frac{q + i\sigma}{q}\right)^{l+1} h_l[(q + i\sigma)r] \exp[-\alpha r]. \tag{34}$$

The integral (34) is simply related to the integral $I^l_\lambda(k/2, q+i\sigma)$ which has been studied in the Appendix A; it can be written as

$$\mathscr{I}^l_\alpha\left(\frac{k}{2}, q + i\sigma\right) = i \frac{(k/2)^l}{q^{l+1}} \int_{\sigma+\alpha}^\infty \mathrm{d}\beta \frac{H^l_\alpha(q; \beta, \sigma)}{(k/2)^2 - (q + i\beta)^2}, \tag{35}$$

from which it is straightforward to see its singularities.

By proceeding in the same way as for the S-waves, we go to recognize the analytic properties of $A_l(q)$ from (33). The $S_l(q)$ function is analytic everywhere in the first sheet of the energy plane, apart from a cut for $q^2 \geqslant 0$, and a branch point in $q^2 = -\mu^2/4$; the last singularity, however, disappears in the matrix element just like in the $l = 0$ (see also Appendix C).

Besides these singularities, other singularities appear in (31): those which come from the integrals $\mathscr{I}$; they have the singularities explicitly shown in the denominator $(q+i\beta)^2 - (k/2)^2$ in (35) ([11]).

([11]) Remember that the singularities from the pole in $(q+i\beta)=0$ of H do not give singularities to the integral $\mathscr{I}$.

It is thus recognized that $A_l(q)$ is a well behaved function in the first sheet of the energy plane, apart from the obvious branch point in $q^2 = 0$ and from the singularities deriving from the integrals $\mathscr{I}$, which give branch points in $(q \pm k/2)^2 = -\alpha^2$ and branch lines in $(q - k/2)^2 \leqslant -(\alpha+\mu)^2$ and $(q+k/2)^2 \leqslant -(\alpha+\mu)^2$. We shall see that this is exactly the same analytic behaviour which is given by a Mandelstam representations with anomalous thresholds.

5. – Analytic properties from perturbation arguments.

5·1. – We discuss the disintegration of a scalar bound state in two scalar particles due to a γ-ray. We call p_1 and p_2 the 4-momenta of the incoming particle (« deuteron ») and photon respectively; q_1 and q_2 those of the final « neutron » and « proton » (see Fig. 1).

The three fundamental relativistic invariant variables we shall use in the following are defined by [12]

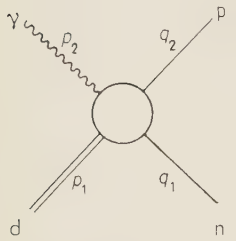


Fig. 1.

$$\left\{\begin{aligned} s &= (p_1 + p_2)^2 = (q_1 + q_2)^2 , \\ t &= (p_1 - q_1)^2 - (p_2 - q_2)^2 , \\ \bar{t} &= (p_1 - q_2)^2 - (p_2 - q_1)^2 , \end{aligned}\right. \tag{36}$$

with

$$s + t + \bar{t} = M^2 + 2m^2 .$$

In the center of mass system we have defined $\boldsymbol{k}$ as the 3-momentum of the photon, $\boldsymbol{q}$ the momentum of the proton, and $\boldsymbol{q}\cdot\boldsymbol{k} = qk\cos\theta$. Those variables are related to s, t, $\bar{t}$, by

$$\left\{\begin{aligned} s &= 4(m^2 + q^2) = (k + E_d)^2 , \\ t &= m^2 - 2kE\left(1 - \frac{q}{E}\cos\theta\right), \\ \bar{t} &= m^2 - 2kE\left(1 + \frac{q}{E}\cos\theta\right), \end{aligned}\right. \tag{37}$$

where

$$E_d = \sqrt{M^2 + k^2} \qquad \text{and} \qquad E = \sqrt{m^2 + q^2} .$$

---

[12] We use the definition of the scalar product $pq = p_0 q_0 - \boldsymbol{p}\cdot\boldsymbol{q}$.

5·2. – We shall attempt now to write a Mandelstam representation for the matrix element $F(s, t)$ describing the deuteron photodisintegration. In order to establish such a double integral representation, we appeal to perturbation theory in the lowest orders as a guide for obtaining the location of singularities. Obviously we have no hamiltonian grounds on which to draw our perturbation diagrams; a justification can only be given *a posteriori* by looking to the results given by the first quantization potential methods. Anyway if we admit that composite particles can be treated in principle as « elementary particles » which are source of other particles (the deuteron as source of nucleons), we draw the lowest order graphs for photodisintegration (we retain always only first order in e) (Fig. 2).

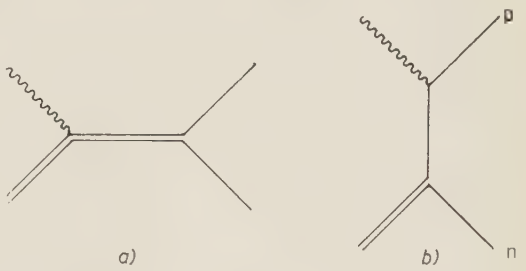


Fig. 2. – ~~~ γ line; —— nucleon line; : deuteron line; – – – pion line.

One sees immediately that these two graphs give a pole in $s = M^2$ and another pole in $t = m^2$ in the matrix element. The crossed diagram of (b) (which would produce a pole in $\bar{t} = m^2$) does not appear, because of the lack of charge of the second outgoing particle (« neutron »).

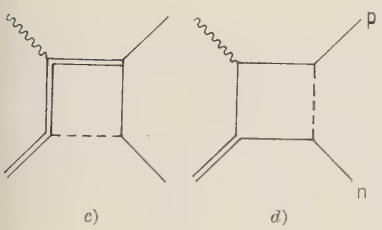


Fig. 3.

Let us examine the following next order diagram (Fig. 3) (*).

In order to investigate the analytic behaviour of the matrix element relative to those graphs we briefly recall the general results given by KARPLUS, SOMMERFIELD and WICHMANN [3].

Let us consider the perturbative diagram of Fig. 4.

One can define the six quantities

$$(38)\qquad \begin{cases} y_{ij} = \dfrac{m_i^2 + m_j^2 - p_{ij}^2}{2m_i m_j}, \\ y_s = \dfrac{m_2^2 + m_4^2 - s}{2m_2 m_4}, \\ y_t = \dfrac{m_1^2 + m_3^2 - t}{2m_1 m_3}. \end{cases}$$

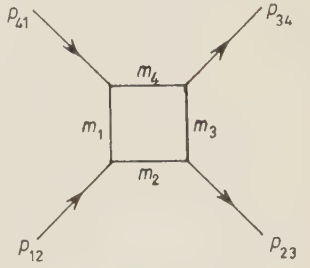


Fig. 4.

(*) We recall that only neutral pions are considered.

If stability conditions are fulfilled for every particle (external or internal) the are subjected to $y_{ij} > -1$, $y_{ij} \leqslant 1$ (the equal sign happens only for γ lines). Then one can put $y_{ij} = \cos\theta_{ij}$. Suppose now every $y_{ij} > 0$. Then the analyticity region in y_s and y_t is $y_s > -1$, $y_t > -1$; this corresponds to $s < (m_2 + m_4)^2$ and $t < (m_1 + m_3)^2$. In this case the branching points have a clear absorptive origin, since the imaginary part of the matrix element is different from zero only when intermediate processes conserving the energy are possible; in this sense one can then speak about regular thresholds.

If, on the contrary, some of the y_{ij} are < 0, it may happen that some of the four sums $y_{ij} + y_{jk}$ be < 0; then the analyticity region is defined by

$$y_t > L_t\,, \qquad y_s > L_s\,, \tag{39}$$

where

$$L_t = \mathrm{Max}\,[L_2, L_4]\,, \qquad L_s = \mathrm{Max}\,[L_1, L_3]$$

and

$$L_1 = \cos(\theta_{23} + \theta_{34}), \quad \text{if} \quad y_{23} + y_{34} < 0\,,$$

$$L_1 = -1 \qquad \text{otherwise}\,,$$

(cyclic permutations define the remaining L's).

In this case the branch points in s and t are lower than the regular ones. They give rise to « anomalous » thresholds. This can happen when some vertex appears in which the « binding energy », $\varepsilon_{ij} = m_i + m_j - M_{ij}$ is little with respect to the involved masses. Moreover, if $\sum \theta_{ij} > 2\pi$, the boundary of the analyticity region is much more complicated. MANDELSTAM has shown [7] that in this case (opposite to the preceding ones) it is not possible to write directly a double dispersion representation.

5·3. – By applying these considerations it is immediately recognized that diagram (*c*) admits a Mandelstam representation with regular absorptive thresholds. In the case of diagram (*d*) we have

$$y_{12} = -1 + \frac{2\varepsilon}{m}\,; \quad y_{23} \quad y_{31} = \frac{\mu}{2m}\,; \quad y_{21} - 1\,,$$

where ε is the binding energy of the deuteron, and higher order terms in ε/m have been neglected (in what follows this will be done whenever possible). We are in the case in which the branch points are given by (39). Thus the threshold for the variable s is regular, whereas the cut in t is anomalous. It starts from t_0, which is given by

$$t_0 = m^2 + 2\mu^2 + 4\alpha\mu\,, \qquad \alpha^2 = \varepsilon m\,. \tag{40}$$

We will assume now that a Mandelstam representation holds not only for the lowest perturbative diagrams, but in general for the matrix element of photodisintegration, and that the thresholds of the double spectral representation are given by diagram (*d*). It seems in fact rather reasonable to suppose that higher order diagrams do not give lower thresholds than (*d*). Under these assumptions one can write

$$F(s,t) = \frac{a}{s-M^2} + \frac{b}{t-m^2} + \int_{4m^2}^{\infty} \mathrm{d}s' \int_{t_0}^{\infty} \mathrm{d}t' \, \frac{M(s',t')}{(s'-s)(t'-t)} \,. \tag{41}$$

a and b are proportional to eN, where e is the electric charge and N is the renormalized coupling constant of the d-$N'N$ vertex. One can interpret N in a non-relativistic way as the normalization factor of the deuteron wave function.

In order to compare this representation with the results obtained by a non-relativistic potential model, we want to extract from (41) the information about the singularities of the fixed angular momentum matrix elements, A_l, by using the formula

$$A_l(q) = \int_{-1}^{1} F(q, \cos\theta) P_l(\cos\theta) \, \mathrm{d}\cos\theta \,. \tag{42}$$

We go to derive A_l from (41) in a region of classical kinematics. The relation between k and q becomes

$$k + \frac{k^2}{2M} - \varepsilon = \frac{q^2}{m} \,. \tag{43}$$

and if one neglects k/m with respect to 1, (43) goes into

$$q^2 = mk - \alpha^2 \,,$$

then

$$t = m^2 - 2\alpha^2 - 2\left(\boldsymbol{q} - \frac{\boldsymbol{k}}{2}\right)^2 .$$

Let us put

$$t' = m^2 + 2\sigma^2 + 4a\sigma \,,$$

$$s' = 4q'^2 + M^2 + 4\alpha^2 \,,$$

then (41) becomes

$$(44)\qquad F(q,\cos\vartheta)=\frac{a/4}{q^2+\alpha^2}-\frac{b/2}{\alpha^2+q^2+(k/2)^2-qk\cos\theta}+$$
$$+2\int_0^\infty\frac{\mathrm{d}q'^2}{q'^2-q^2}\int_\mu^\infty\mathrm{d}\sigma\,\frac{(\sigma+\alpha)M(\sigma,q'^2)}{(\sigma+\alpha)^2+q^2+(k/2)^2-qk\cos\theta}\,.$$

Therefore, denoting as Q_l the second solution of the Legendre equation we obtain from (42)

$$(45)\qquad A_l(q)=\frac{a/2}{q^2+\alpha^2}\delta_{l,0}-\frac{b}{qk}Q_l\left(\frac{\alpha^2+q^2+k^2/4}{qk}\right)+$$
$$+4\int_0^\infty\frac{\mathrm{d}q'^2}{q'^2-q^2}\int_\mu^\infty\mathrm{d}\sigma\,\frac{(\sigma+\alpha)M(\sigma,q'^2)}{qk}Q_l\left(\frac{(\sigma+\alpha)^2+q^2+k^2/4}{qk}\right).$$

It is easily seen that the analytic properties of the fixed angular momentum matrix elements given by the Mandelstam representation coincide with those obtained by potential methods. Namely one finds immediately logarithmic branch points in $(q\pm k/2)^2=-\alpha^2$ and branch lines in $(q-k/2)^2\leqslant-(\alpha+\mu)^2$ and $(q+k/2)^2\leqslant-(\alpha+\mu)^2$, which originate from the singularities in the momentum transfer t, and a cut on the real positive axis of q^2 which originates from singularities in the variable s. Moreover, in the case $l=0$, (45) shows the pole of the deuteron intermediate state, which is also reproduced in the $l=0$ matrix element in the potential approach.

6. – Concluding remarks.

We have shown that in a two body photodisintegration process (the photodisintegration of the deuteron) some consequences of the Mandelstam representation are reproduced in a non-relativistic potential model. Indeed the analytic properties of the fixed angular momentum matrix elements, derived from the double integral representation with anomalous thresholds, are the same as obtained by potential theory. The use of wave functions in potential theory gives rise only to anomalous thresholds; it is clear that the agreement between a potential model and a complete relativistic Mandelstam representation breaks down when physical absorptive thresholds appear; in fact the potential obviously cannot give account of the production of real particles.

Though this is not yet a demonstration of Mandelstam representation by potential models, in processes involving bound states, it is a strong argu-

ment in favour of the general validity of the representation with anomalous thresholds in the deuteron photodisintegration case. Following the potential theory the anomalous thresholds given by perturbation methods have clearly their origin from a structure effect of the deuteron, which is a loosely bound composite particle.

It seems therefore possible to apply the most recent tools suggested by field theory, as double dimensional integral representations, to the nuclear physics; at low energy they should give the same results as the classical potential methods.

* * *

We are deeply indebted to Prof. S. FUBINI for his constant advice and help, and for illuminating discussions during the course of this work. We with also to thank Prof. A. MARTIN, for fruitful discussions on many points of our work, and Prof. R. STROFFOLINI for his kind interest and useful comments.

APPENDIX A

We want to evaluate the integrals of the type

$$I_\nu^l(k, q + i\sigma) = \int_0^\infty \mathrm{d}r\, r \exp[-\nu r] j_l(kr) h_l[(q + i\sigma)r] . \tag{A.1}$$

We put then

$$\frac{\exp[-\nu r]}{r} h_l[(q + i\sigma)r] = \int_0^\infty \mathrm{d}\beta \left(\frac{q + i\beta}{q + i\sigma}\right)^{l+1} h_l[(q + i\beta)r] H_\nu^l(q; \beta, \sigma) . \tag{A.2}$$

In order to calculate the kernel $H_\nu^l(q; \beta, \sigma)$ we multiply (A.2) by $r^2 j_l(kr)$ and integrate over r; then we obtain

$$I_\nu^l(k, q + i\sigma) = \int_0^\infty \mathrm{d}r\, r \exp[-\nu r] j_l(kr) h_l[(q + i\sigma)r] = \tag{A.3}$$

$$= \int_0^\infty \mathrm{d}r \int_0^\infty \mathrm{d}\beta\, r^2 j_l(kr) h_l[(q + i\beta)r] \left(\frac{q + i\beta}{q + i\sigma}\right)^{l+1} H_\nu^l(q; \beta, \sigma) .$$

The second integral appearing in (A.3) is immediately calculated by interchanging the order of integration and remembering that

$$\int_0^\infty r^2 j_l(\alpha r) h_l(\beta r)\,\mathrm{d}r = -i\,\frac{\alpha^l}{\beta^{l+1}}\,\frac{1}{\alpha^2-\beta^2}\,. \tag{A.3'}$$

One then obtains

$$I_\nu^l(k,\,q+i\sigma) = -i\,\frac{q^l}{(q+i\sigma)^{l+1}}\int_0^\infty \mathrm{d}\beta\,\frac{H_\nu^l(q;\,\beta,\,\sigma)}{k^2-(q+i\beta)^2}\,. \tag{A.4}$$

The first integral in (A.3) can be written

$$I_\nu^l(k,\,q+i\sigma) = -\int_0^\infty \mathrm{d}r\,\frac{\mathrm{d}}{\mathrm{d}\nu}\exp\left[-\nu r\right] j_l(kr)\,h_l[(q+i\sigma)r]\,. \tag{A.5}$$

We use now for $h_l[(\varphi+i\sigma)r]$ the integral representation

$$h_l[(q+i\sigma)r] = \frac{(-i)^{l+1}}{q+i\sigma}\int_\sigma^\infty \mathrm{d}\gamma\,P_l\left(\frac{q+i\gamma}{q+i\sigma}\right)\exp\left[(iq-\gamma)r\right].$$

Then (A.5) goes into

$$I_\nu^l(k,\,q+i\sigma) = \tag{A.6}$$

$$= -\frac{(-i)^{l+1}}{q+i\sigma}\int_0^\infty \mathrm{d}r\int_0^\infty \mathrm{d}\gamma\,\vartheta(\gamma-\sigma)P_l\left(\frac{q+i\gamma}{q+i\sigma}\right)\frac{\mathrm{d}}{\mathrm{d}\nu}\exp\left[(iq-\gamma-\nu)r\right] j_l(kr)\,,$$

where $\vartheta(\gamma-\sigma)$ is the step function

$$\vartheta(\gamma-\sigma) = \begin{cases} 0 & \text{for } \gamma<\sigma,\\ 1 & \text{for } \gamma\geqslant\sigma. \end{cases}$$

In (A.6) we may interchange the order of integration and get

$$I_\nu^l(k,\,q+i\sigma) = -\frac{(-i)^{l+1}}{q+i\sigma}\int_0^\infty \mathrm{d}\gamma\,\vartheta(\gamma-\sigma)\,P_l\left(\frac{q+i\gamma}{q+i\sigma}\right)\frac{\mathrm{d}}{\mathrm{d}\nu}\,J_{\nu+\gamma}^l(k,\,q)\,, \tag{A.7}$$

where we have put

$$J_\tau^l(k,\,q) = \int_0^\infty \mathrm{d}r\,j_l(kr)\exp\left[(iq-\tau)r\right]. \tag{A.8}$$

By introducing for j_l the integral representation

$$j_l(kr) = \frac{1}{2i^l}\int_{-1}^{1} \mathrm{d}x\, P_l(x)\, \exp\,[ikrx]\,, \tag{A.9}$$

(A.8) goes into

$$J_\tau^l(k, q) = \frac{1}{2i^l}\int_0^\infty \mathrm{d}r \int_{-1}^{1} \mathrm{d}x\, P_l(x)\, \exp\,[[i(kx + q) - \tau]r\,]. \tag{A.10}$$

In (A.10) we can now interchange the order of integration and integrate over r: the result is

$$J_\tau^l(k, q) = \frac{1}{2i^l}\int_1^1 \mathrm{d}x \frac{P_l(x)}{\tau - i(kx + q)}\,. \tag{A.11}$$

Making use of the elementary identity

$$\frac{1}{A + B} = \frac{1}{A} - \frac{B}{A^2} + \ldots + (-)^l \frac{B^l}{A^l(A + B)}\,,$$

the integral (A.11) can be rewritten in the form

$$J_\tau^l(k, q) = \frac{i^{l-1}}{2}\int_{-1}^{1} \mathrm{d}x \frac{k^l}{(q + i\tau)^l} \frac{x^l P_l(x)}{kx + q + i\tau}\,. \tag{A.12}$$

Let us now make a first substitution, $x = 1/u$; the integral (A.12) then becomes

$$J_\tau^l(k, q) = i^{l-1}\int_1^\infty \frac{\mathrm{d}u}{u} \frac{k^l}{[(q + i\tau)u]^{l-1}} \frac{P_l(1/u)}{k^2 - [(q + i\tau)u]^2}\,. \tag{A.13}$$

By introducing now a new substitution, $(q + ia)u = q + i\beta$, we finally obtain

$$J_\tau^l(k, q) = -\, i^l \int_0^\infty \mathrm{d}\beta\, e^{(\beta - \tau)} \frac{k^l}{(q + i\beta)^l} \frac{P_l((q + i\tau)/(q + i\beta))}{k^2 - (q + i\beta)^2}\,. \tag{A.14}$$

In going from (A.12) to (A.14) the original path of integration has been distorted so to coincide with the real β axis, which is of course possible for real q in account of the Cauchy theorem.

One could think from formula (A.14) that there are further singularities from the denominators in $(q + i\beta)$; however, due to the peculiar form of the

Legendre polinomials, those singularities are only apparent, as one can see from (A.11) or from direct calculation.

By means of (A.14), (A.7) gives

$$(A.15)\qquad I_\nu^l(k, q+i\sigma) = -i\frac{k^l}{q+i\sigma}\int_0^\infty \frac{\mathrm{d}\beta}{k^2-(q+i\beta)^2}\int_0^\infty \mathrm{d}\gamma\, \ell(\gamma-\sigma)\cdot$$

$$\cdot\frac{1}{(q+i\beta)^l} P_l\left(\frac{q+i\gamma}{q+i\sigma}\right)\frac{\mathrm{d}}{\mathrm{d}\nu} P_l\left(\frac{q+i(\gamma+\nu)}{q+i\beta}\right)\vartheta(\beta-\nu-\gamma)\,,$$

where the order of integration has been interchanged (which is obviously possible).

By equating now (A.4) and (A.15) one has the following expression for the kernel H:

$$(A.16)\qquad H_\nu^l(q;\beta,\sigma) = -\left(\frac{q+i\sigma}{q+i\beta}\right)^l\int_0^\infty \mathrm{d}\gamma\, \ell(\gamma-\sigma) P_l\left(\frac{q+i\gamma}{q+i\sigma}\right)\cdot$$

$$\cdot\frac{\mathrm{d}}{\mathrm{d}\nu} P_l\left(\frac{q+i(\gamma+\nu)}{q+i\beta}\right)\ell(\beta-\nu-\gamma)\,,$$

or, performing the derivative,

$$(A.17)\qquad H_\nu^l(q;\beta,\sigma)\quad \left(\frac{q+i\sigma}{q+i\beta}\right)^l \vartheta(\beta-\nu-\sigma)\left\{P_l\left(\frac{q+i(\beta-\nu)}{q+i\sigma}\right)-\right.$$

$$\left.-\frac{i}{q+i\beta}\int_\sigma^{\beta-\nu}\mathrm{d}\gamma P_l\left(\frac{q+i\gamma}{q+i\sigma}\right)P_l\left(\frac{q+i(\gamma+\nu)}{q+i\beta}\right)\right\}.$$

From (A.17) it is easily seen that $H_\nu^l(q;\beta,\sigma)$ is a limited function of real σ and β (we remember that in (A.16) appears $\vartheta(\beta-\nu-\sigma)$, which shows that $\sigma\to\infty$ implies $\beta\to\infty$).

Now, introducing (A.17) into (A.15), the integral $I_\nu^l(k, q+i\sigma)$ can be written

$$(A.18)\qquad I_\nu^l(k, q+i\sigma) = i\frac{k^l}{(q+i\sigma)^{l+1}}\int_{\sigma+\nu}^\infty \mathrm{d}\beta\frac{H_\nu^l(q;\beta,\sigma)}{k^2-(q+i\beta)^2}\,.$$

Appendix B

With the present appendix we will support the unrigorous assumption we have made to write the relation (30) in the text. First we shall show that the definition of the S matrix is not meaningless, *i.e.* that the integrals

$$(B.1)\qquad R = \int_\mu^\infty \varrho(q,\sigma)\,\mathrm{d}\sigma\,,$$

are convergent. In other words we have to show that the ϱ's can be written as

$$\varrho(q,\sigma) = \delta(\sigma) + \frac{G(q,\sigma)}{(q+i\sigma)^2}, \tag{B.2}$$

where $G(q,\sigma)$ is a limited function of real $\sigma > 0$. In order to do this we appeal to the integral eq. (23) for the ϱ's:

$$\varrho(q,\sigma) = \delta(\sigma) + f\frac{1}{\sigma(\sigma-2iq)}\int_0^{\sigma-\mu} \mathrm{d}\beta\, \varrho(q,\beta) H_\mu^l(q;\sigma,\beta), \tag{B.3}$$

which may be solved in the familiar way by the series

$$\varrho(q,\sigma) = \delta(\sigma) + \sum_1^\infty{}_n \varrho_n(q,\sigma),$$

where

$$\begin{cases} \varrho_1 \quad (q,\sigma) = fA(\sigma) H_\mu^l(q;\sigma,0)\theta(\sigma-\mu), \\ \varrho_{n+1}(q,\sigma) = fA(\sigma)\int_0^{\sigma-\mu} \mathrm{d}\beta\, H_\mu^l(q;\sigma,\beta)\varrho_n(q,\beta), \end{cases} \tag{B.4}$$

where we have put $[\sigma(\sigma-2iq)]^{-1} = A(\sigma)$.

By remembering that in App. A the kernel H has been shown to be a limited function of σ and β (if $0 < \beta \leqslant \sigma - \mu \leqslant \infty$) so that

$$|H_\mu^l(q;\sigma,\beta)| \leqslant M; \qquad M = M(q), \tag{B.5}$$

it follows by induction that

$$|\varrho_{n+1}(q,\sigma)| \leqslant f^{n+1}|A(\sigma)|\,\frac{M^{n+1}}{n!}\left\{\int_\mu^{\sigma-\mu}|A(\beta)|\,\mathrm{d}\beta\right\}^n. \tag{B.6}$$

Consequently the series converges to the required solution. By inspection of (B.6) the validity of (B.2) with $|G(q,\sigma)| \leqslant N$; $N = N(q)$, follows.

From (B.2) it is straightforward to see that, by putting

$$I(r) = r^{l+1}\int_\mu^\infty (q+i\sigma)^{l+1}\varrho(q,\sigma) h_l[(q+i\sigma)r]\,\mathrm{d}\sigma,$$

then

$$\lim_{r\to 0} I(r) = \mathrm{const}\int_\mu^\infty \varrho(q,\sigma)\,\mathrm{d}\sigma. \tag{B.9}$$

We start by noting that:

$$h_l[(q + i\sigma)r] = \sum_1^{l+1} a_k \frac{\exp[-(\sigma - iq)r]}{(q + i\sigma)^k r^k} .$$

it follows

$$I(r) - \sum_1^{l-1} a_k I_k(r) ,$$

where

$$I_k(r) = \int_\mu^\infty [(q + i\sigma)r]^{l+1-k} \varrho(q, \sigma) \exp[-(\sigma - iq)r] \, d\sigma .$$

In particular

$$I_{l+1}(r) = \int_\mu^\infty \varrho(q, \sigma) \exp[-(\sigma - iq)r] \, d\sigma ,$$

and

$$\lim_{r \to 0} I_{l-1}(r) = \int_\mu^\infty \varrho(q, \sigma) \, d\sigma .$$

By reminding that if $f(\sigma) \underset{\sigma \to \infty}{\sim} \sigma^n$; $(n > -1)$, then its Laplace transform

$$\int_\mu^\infty f(\sigma) \exp[-\sigma r] \, d\sigma \underset{r \to 0}{\sim} r^{-n-1} ,$$

it follows immediately

$$|\lim_{r \to 0} I_k(r)| = \lim_{r \to 0} r^{l+1-k} \int_\mu^\infty (q + i\sigma)^{l-k-1} G(q, \sigma) \exp[-\sigma r] \, d\sigma = 0 ,$$

for $k = 1, 2, \ldots, l-1$.

Finally, by means of the Schwartz inequality it is easy to show that $\lim_{r \to 0} I_l(r) = 0$. Indeed one has

$$|\lim_{r \to 0} I_l(r)| = \lim_{r \to 0} r \left| \int_\mu^\infty (q + i\sigma)^{-1} G(q, \sigma) \exp[-\sigma r] \, d\sigma \right| \leq$$

$$\leq \lim_{r \to 0} r \left\{ \int_\mu^\infty \frac{d\sigma}{q^2 + \sigma^2} \int_\mu^\infty |G(q, \sigma)|^2 \exp[-2\sigma r] \, d\sigma \right\}^{\frac{1}{2}} = 0 .$$

The validity of (B.7) then follows.

Appendix C

We shall show here that (in first perturbative order) the singularities of the S matrix in $q = i\sigma_0/2$ ($\sigma_0 > \mu$) are not effective in the expression of $\psi_l(q, r)$.

We refer, for simplicity, instead than to ψ_l, to the socalled regular solution, *i.e.* to

$$R(q)\psi_l(q, r) = R_l(q)R_l(-q, r) - R_l(-q)R_l(q, r)\,,$$

which has the same analytic properties in $\mathrm{Im}\, q > 0$, apart from the bound state poles coming from $R_l(q) = 0$.

By using for $R_l(q, r)$ and $R_l(q)$ the spectral representations (25) and (29), and introducing for ϱ the first approximation in the coupling constant f, one has

$$R_l(q)\psi_l(q, r) = -\frac{(2l-1)!!}{q^{l+1}}\left\{\left(1 + f\int_\mu^\infty d\sigma\,\frac{H^l_\mu(q;\sigma,0)}{\sigma(\sigma-2iq)}\right)\cdot\right.$$

$$\cdot\left(h_l(-qr) + f\int_\mu^\infty d\sigma\left(\frac{-q+i\sigma}{-q}\right)^{l+1} h_l[(-q+i\sigma)r]\,\frac{H^l_\mu(-q;\sigma,0)}{\sigma(\sigma+2iq)}\right) -$$

$$\left. -(-)^{l+1}\left(1 + f\int_\mu^\infty d\sigma\,\frac{H^l_\mu(-q;\sigma,0)}{\sigma(\sigma+2iq)}\right)\left(h_l(qr) + f\int_\mu^\infty d\sigma\left(\frac{q+i\sigma}{q}\right)^{l-1} h_l[(q-i\sigma)r]\,\frac{H^l_\mu(q;\sigma,0)}{\sigma(\sigma-2iq)}\right)\right\},$$

where one may easily see the singular terms for $q = i\sigma_0/2$. We collect them together (retaining only first order in f)

$$-f\,\frac{(2l-1)!!}{(-i(\sigma_0/2))^{l+1}}\left\{\int_\mu^\infty d\sigma\left(\frac{2\sigma-\sigma_0}{\sigma_0}\right)^{l+1} h_l\left[i\left(\sigma-\frac{\sigma_0}{2}\right)r\right] H^l_\mu\!\left(\begin{matrix} i(\sigma_0/2);\ \sigma, 0 \\ \sigma(\sigma-\sigma_0)\end{matrix}\right)\right.$$

$$\left. - h_l\left(i\,\frac{\sigma_0}{2}\,r\right)\int_\mu^\infty d\sigma\,\frac{H^l_\mu(-i(\sigma_0/2);\sigma,0)}{\sigma(\sigma-\sigma_0)}\right\},$$

and it is obvious to observe that in $\sigma = \sigma_0$ the preceding quantity is regular.

RIASSUNTO

In questo lavoro vengono studiate le proprietà analitiche dell'ampiezza d'urto di fotodisintegrazione del deutone a fissato momento angolare, mediante un modello di teoria del potenziale. Si dimostra che esiste un completo accordo tra le proprietà così ottenute e quelle derivanti dalla rappresentazione di Mandelstam, nel caso in cui siano presenti soglie anomale, ricavate con argomenti perturbativi. Si ha così un argomento in favore della validità generale delle tecniche dispersive, anche in fisica nucleare.

Dispersion Relations for some Photon Reactions.

A. Minguzzi and R. F. Streater

CERN - Geneva

(ricevuto il 1° Agosto 1960)

Summary. — By using Dyson's integral representation directly for the absorptive part of the T matrix, certain dispersion relations involving photons and nuclei are proved more simply than by the method of Bogoliubov *et al.* The range of validity is always less than that obtained by the best possible techniques, but fewer physical assumptions are needed. In particular the existence of an interpolating field associated with light nuclei is not needed. In the usual e^2 approximation, we can treat photoproduction, Compton scattering and bremsstrahlung from nuclei.

Introduction.

We study the matrix element

$$(1) \quad T(p_1, p_2, k_1 + k_2) = \int \langle p_1 | \theta(x) \left| j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right)\right| | p_2 \rangle \exp\left[i \frac{k_1 + k_2}{2} x \right] \mathrm{d}^4 x \,,$$

which, owing to its retarded nature, is a regular function of $q = (k_1 + k_2)/2$ in the forward tube $\mathscr{T}(q)$, *i.e.* if $\mathrm{Im}\, q$ is a forward timelike vector. This information leads to analyticity in the cut energy plane (and hence to as dispersion relation) only for suitable negative values of k^2. Instead of following the continuation procedure of Bogoliubov [1], here we add the mass-spectral conditions directly to the forward tube. This problem was solved by the

[1] N. N. Bogoliubov and D. V. Shirkov: *Introduction to the Theory of Quantized Fields* (New York, 1959); H. Lehmann: *Suppl. Nuovo Cimento*, **14**, 153 (1959); H. J. Bremermann, R. Oehme and J. G. Taylor: *Phys. Rev.*, **109**, 2178 (1958).

Jost-Lehmann-Dyson representation ([3-5]). We find that the domain of regularity can immediately be shown to be the cut energy plane only in cases where the photon is involved, and in the usual approximation in which all intermediate states containing photons are ignored. The method proves dispersion relations for the process $\gamma+\mathrm{n}\to\gamma+\mathrm{n}$ up to $\Delta^2=.8\mu^2$, for the process $\gamma+\mathrm{n}\to\pi+\mathrm{n}$ for Δ^2 at the threshold, for the process $\gamma+\mathrm{n}\to\gamma+\gamma+\mathrm{n}$ for a suitable momentum transfer between the nucleons, and a suitable configuration of the final boson system, and for γ-nucleus elastic scattering for light nuclei (*i.e.* provided $(Z/\sqrt{137})^2\ll 1$), for momentum transfers of the order of the binding energy of the least tightly bound nucleon. This can be shown without the necessity of ascribing an interpolating field ([6]) to the nucleus, which remains in the state vector throughout the proof.

In order to prove dispersion relations, we must eliminate the contributions from the one-particle intermediate states, which give rise to poles at, say, $(\pm q+(p_1+p_2)/2)^2=M^2$. This is done by considering, instead of $T(p_1p_2q)$, the function

$$\left[\left(q+\frac{p_1+p_2}{2}\right)^2-M^2\right]\left[\left(-q+\frac{p_1+p_2}{2}\right)^2-M^2\right]T(p_1,p_2,q)\equiv\tilde{T}\,, \tag{2}$$

which does not contain this contribution, but otherwise has the same support and spectrum. Then $T(\omega)$ has the same domain at $\tilde{T}(\omega)$ except for poles at $(\pm\omega+E_1)^2-\omega^2+\Delta^2=M^2$.

1. – Photon-nucleon elastic scattering ([2]).

We introduce $\tilde{T}$, given by (1) and (2) where M is the nucleon mass and $J(x)$ the photon current, and apply the Jost-Lehmann ([3]) representation with $m_1=m_2=M+\mu$, ($\mu=$ meson mass) in the Breit system $\boldsymbol{p}+\boldsymbol{p}'=0$. We get

$$\tilde{T}=\frac{1}{2\pi}\int \mathrm{d}^3u\,\mathrm{d}k^2\,\frac{\Phi_1(\boldsymbol{u},k)+\dfrac{k_{10}+k_{20}}{2}\,\Phi_2(\boldsymbol{u},k)}{k^2+\left(\dfrac{\boldsymbol{k}_1+\boldsymbol{k}_2}{2}-\boldsymbol{u}\right)^2-\left(\dfrac{k_{10}+k_{20}}{2}\right)^2}\,. \tag{3}$$

([2]) R. Oehme and J. G. Taylor: *Phys. Rev.*, **113**, 371 (1959).
([3]) R. Jost and H. Lehmann: *Nuovo Cimento*, **5**, 1598 (1957).
([4]) F. J. Dyson: *Phys. Rev.*, **110**, 1460 (1951).
([5]) R. F. Streater: *Garding's proof of the Jost-Lehmann-Dyson representation*, CERN Report 8483/TH. 89.
([6]) W. Zimmermann: *Nuovo Cimento*, **10**, 597 (1958); K. Nishijima: *Phys. Rev.*, **111**, 995 (1958).

By defining

$$\omega = \frac{(k_1 + k_2)(p_1 + p_2)}{2\sqrt{(p_1 + p_2)^2}} , \tag{4}$$

we have

$$\frac{K_{10} + K_{20}}{2} = \omega , \qquad \boldsymbol{p}_1 = -\boldsymbol{p}_2 , \qquad p_{1.0} = p_{2.0} = \sqrt{\Delta^2 + \boldsymbol{M}^2} \equiv E_1 ,$$

and the support of $\varphi_{1,2}$ is

$$k \geqslant \max\{0, M + \mu - \sqrt{E_1^2 - \boldsymbol{u}^2}\} , \qquad 0 \leqslant u^2 \leqslant E_1 . \tag{5}$$

By choosing a polar axis in the scattering plane perpendicular to $\boldsymbol{p}$ we get

$$\widetilde{T} = \frac{1}{2\pi}\int u^2\,\mathrm{d}u \int_{-1}^{1} \mathrm{d}(\cos\theta) \int_0^{2\pi} \mathrm{d}\varphi\ \mathrm{d}k^2 \cdot \tag{6}$$

$$\cdot \frac{\Phi_1(\sin\theta, \cos\varphi\,|\,\boldsymbol{u}\,|\,k) + \omega\ \Phi_2(\sin\theta, \cos\varphi\,|\,\boldsymbol{u}\,|\,k)}{k^2 - \Delta^2 + \boldsymbol{u}^2 - 2\,|u|\cos\theta\sqrt{\omega^2 - \Delta^2}} ,$$

which can be written

$$\widetilde{T} = \frac{1}{2\pi}\int u^2\,\mathrm{d}u\,\mathrm{d}k^2 \int_0^1 \mathrm{d}(\cos\theta) \int_0^{2\pi} \mathrm{d}\varphi\ \frac{[\Phi_1 + \omega\,\Phi_2]\,2\,[k^2 - \Delta^2 - \boldsymbol{u}^2]}{(k^2 - \Delta^2 + \boldsymbol{u}^2)^2 - 4\boldsymbol{u}^2\cos^2\theta(\omega^2 - \Delta^2)} . \tag{7}$$

If the smallest value of $k^2 - \Delta^2 + \boldsymbol{u}^2$, (as k, u vary over the support of $\Phi_{1,2}$) is positive, we can prove dispersion relations. Otherwise, no point in the ω-plane is a regular point. The latter case holds for all processes not involving photons (*). For $\gamma + \mathrm{n} \to \gamma + \mathrm{n}$, however, $\min(k^2 - \Delta^2 + \boldsymbol{u}^2)$ is positive, provided that $\Delta^2 < \sim .8\ \mu^2$.

On introducing the two weight functions

$$\widetilde{\Phi}_{1\,2}(\omega') = \int u^2\,\mathrm{d}u\,\mathrm{d}k^2 \int_0^1 \mathrm{d}(\cos\theta) \cdot \tag{8}$$

$$\cdot \int_0^{2\pi} \mathrm{d}\varphi \Phi_{1.2}(2k^2 - 2\Delta^2 + 2\boldsymbol{u}^2)\,\delta[(k^2 - \Delta^2 + u^2)^2 - 4\boldsymbol{u}^2\cos^2\theta(\omega'^2 - \Delta^2)] ,$$

(*) This remark was stressed by J. G. TAYLOR: *Lectures on dispersion relations* University of Maryland, ASTIA no. 202640.

we get

$$\tilde{T} = \frac{1}{2\pi}\int_{\omega_0}^{\infty} \frac{2\tilde{\Phi}_1(\omega')\omega' \,\mathrm{d}\omega'}{\omega'^2 - \omega^2} + \frac{\omega}{2\pi}\int_{\omega_0}^{\infty} \frac{2\tilde{\Phi}_2(\omega')\omega' \,\mathrm{d}\omega'}{\omega'^2 - \omega^2}\,, \tag{9}$$

where

$$\omega_0 = \left[\min\left(\frac{k^2 - \Delta^2 + \boldsymbol{u}^2}{2|\boldsymbol{u}|\cos\theta}\right)^2 + \Delta^2\right]^{\frac{1}{2}} = \frac{-2E_\Delta + m^2 + m_1^2}{2E_\Delta}\,,$$

the « correct » threshold corresponding to $(p+k)^2 = (M+\mu)^2$. Then

$$\tilde{T} = \frac{1}{2\pi}\left(\int_{-\infty}^{-\omega_0} + \int_{\omega_0}^{\infty}\right)\frac{\varepsilon(\omega')[\tilde{\Phi}_1(\omega') + \omega'\tilde{\Phi}_2(\omega')]}{\omega' - \omega}\,\mathrm{d}\omega'. \tag{10}$$

The numerator is exactly $\tilde{A}(\omega')$, the absorptive part, defined as

$$\tilde{A} = \int \mathrm{d}x^4 \exp[iqx]\left[\left(\pm i\frac{\partial}{\partial x} + \frac{p_1 + p_2}{2}\right)^2 - M^2\right]\langle p_1|\left[j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right)\right]|p_2\rangle\,.$$

For a theory invariant under TP, $\tilde{A}(\omega')$ is real and equal to the imaginary part of $\tilde{T}(\omega')$ [7]. For elastic scattering, this follows already from momentum conservation, without the use of TP.

In the same way, neglecting intermediate states with more than one photon, we can show dispersion relations for the process

$$\gamma + \text{nucleus} \to \gamma + \text{nucleus}$$

for $\Delta^2 \leqslant \Delta_0^2 \simeq$ binding energy of the most loosely bound nucleons.

2. – Photo-production from nucleons.

It turns out that dispersion relations can be shown for the largest momentum transfer if we apply the reduction formula to the nucleon field, *i.e.* we consider

$$\tilde{T} = \left[\left(\frac{k_1 + k_2}{2} \pm q\right)^2 - M^2\right]\int \langle\gamma|\theta(x)\left[f\left(\frac{x}{2}\right), f^+\left(-\frac{x}{2}\right)\right]|\pi\rangle \exp[iqx]\,\mathrm{d}^4x\,, \tag{11}$$

$$q = \frac{p_1 + p_2}{2}\,, \qquad k_1^2 = 0\,, \qquad k_2^2 = \mu^2\,.$$

---

[7] R. OEHME: *Phys. Rev.*, **100**, 1503 (1955), Appendix.

Define

$$\omega = \frac{(p_1 + p_2)(k_1 + k_2)}{2\sqrt{(k_1 + k_2)^2}}, \tag{12}$$

and in the Breit system of the bosons ($\boldsymbol{k}_1 + \boldsymbol{k}_2 = 0$)

$$q = \left\{\omega,\ 0,\ \frac{2\omega\mu^2}{4k\overline{w}},\ \sqrt{\omega^2 \frac{\Delta^2}{\overline{k}^2} - m^2 - \Delta^2}\right\},$$

$$k_1 = \left\{\frac{\overline{w}^2 - \mu^2}{2\overline{w}},\ 0,\ k,\ 0\right\},$$

$$k_2 = \left\{\frac{\overline{w}^2 + \mu^2}{2\overline{w}},\ 0,\ -k,\ 0\right\},$$

where

$$\overline{w}^2 = 2\mu^2 + 4\Delta^2, \qquad \overline{k}^2 = \frac{(\overline{w}^2 - \mu^2)^2}{4\overline{w}^2}.$$

Then as above

$$\widetilde{T} = \frac{1}{2\pi}\int \mathrm{d}^3 u\, \mathrm{d}k^2 \cdot \tag{13}$$

$$\cdot \frac{\Phi_1 + q_0\Phi_2}{k^2 + \boldsymbol{u}^2 - \Delta^2 - m^2 + 2|\boldsymbol{u}|\cos\theta\sqrt{\omega^2(\Delta^2/k^2) - m^2 - \Delta^2 + 2|\boldsymbol{u}|\sin\theta\cos\varphi(\omega\mu^2/2k\overline{w})}},$$

where $\Phi_{1,2}$ have support

$$0 \leqslant u^2 \leqslant \frac{\mu^2}{2} + \Delta^2; \qquad k \geqslant \max\left\{0, m + \mu - \sqrt{\frac{\mu^2}{2} + \Delta^2 - \boldsymbol{u}^2}\right\}.$$

As before one can eliminate the square root in (13) by using rotational invariance. The domain of regularity is the cut plane, provided

$$\left|\min\left(\frac{k^2 + \boldsymbol{u}^2 - m^2 - \Delta^2}{2|\boldsymbol{u}|}\right)\right|^2 \geqslant \frac{\frac{1}{4}(\mu^4/k^2\overline{w}^2)(\Delta^2 + m^2)}{1 - \frac{1}{4}(\mu^4/\overline{k}^2\overline{w}^2)}, \tag{14}$$

and one can show that with the actual values of the meson and nucleon masses (14) is satisfied with $\Delta^2 = \Delta^2_{\text{thresh}}$.

3. – Bremsstrahlung from nucleons.

We can prove regularity in the cut plane also for the process

$$\gamma + \mathrm{n} \to \gamma' + \gamma'' + \mathrm{n}'.$$

Consider

$$\tilde{T} \propto \int \langle \gamma | \theta(x) \left[f\left(\frac{x}{2}\right), f^{+}\left(\frac{x}{2}\right) \right] | \gamma' \gamma''^{\text{out}} \rangle \exp\left[i \frac{n'+n}{2} x \right].$$

Define the Breit system by

$$\boldsymbol{\gamma} + \boldsymbol{\gamma}' + \boldsymbol{\gamma}'' = 0$$

and fix the « mass » of the $\gamma'+\gamma''$ system as $\xi^2 = (\gamma'+\gamma'')^2$, and define

$$\omega = \frac{(\gamma+\gamma'+\gamma'')(n+n')}{2\sqrt{(\gamma+\gamma'+\gamma'')^2}},$$

then we get exactly the same expression as in (13) with ξ^2 instead of μ^2, Δ^2 being always the momentum transfer between the nucleons. It follows that ω is expressed in terms of $W^2 = (\gamma+n)^2$, $\Delta^2 = -\frac{1}{4}(n-n')^2$, $\xi^2 = (\gamma'+\gamma'')^2$ only. As the two other invariants we take the polar co-ordinates θ', φ'' of the vector $\boldsymbol{\gamma}'-\boldsymbol{\gamma}''$, one axis being along $\boldsymbol{n}'$ and the polar axis perpendicular to $\boldsymbol{\gamma}'+\boldsymbol{\gamma}''$. Then the kinematics of the problem is specified by the five parameters W^2, Δ^2, ξ^2, φ', θ' (8). One can follow the procedure in (13), proving dispersion relations in ω under condition (14), with μ^2 replaced by ξ^2. There are two modificatiins. In the first place, the square root in the denominator of eq. (13) cannot be removed in this case, since the weight-function is no longer invariant under rotations around the $\boldsymbol{\gamma}$-axis. Secondly the absorptive part is no longer the imaginary part.

One can localise the branch point at the value $W = \pm m+\mu$, $\pm m+2\mu \ldots$ by considering, instead of T, the related functions

$$T^{+} = T(\theta', \varphi') + T(\pi-\theta', \varphi'),$$

$$T^{-} = \frac{T(\theta', \varphi') - T(\pi-\theta', \varphi')}{\sqrt{\omega^2(\Delta^2/\bar{k}^2) - M^2 - \Delta^2}}.$$

4. – Conclusion.

Although, by this method, we cannot show dispersion relations for as large a value of Δ^2 as by the standard method, this method is so much simpler that we can hope to incorporate some of the consequences of unitarity without the work becoming too complicated. The possibility of showing photon disper-

(8) R. Ascoli and A. Minguzzi: *Phys. Rev.*, **118**, 1435 (1960).

sion relations by Dyson's representation is closely related to the possibility [9] of justifying « pology » for some photon processes. As remarked in ref. [9] it is essential that we can treat the electromagnetic forces to order e^2.

---

[9] J. GUNSON and J. G. TAYLOR: *Nuovo Cimento*, **15**, 806 (1960).

RIASSUNTO

In questa nota si dimostrano relazioni di dispersione in cui intervengono fotoni senza usare la tecnica di Bogoliubov, partendo dalla rappresentazione integrale di Dyson del commutatore non ritardato. I valori del « momentum transfer » che si ottengono sono minori di quelli ottenuti con la tecnica migliore; ma il nostro procedimento richiede un minor numero di ipotesi; in particolare non è necessario introdurre « campi interpolanti » per i nuclei. Nell'approssimazione e^2, si provano relazioni di dispersione per fotoproduzione, scattering Compton e bremsstrahlung su nuclei.

Magnetohydrodynamische Schwingungen in einem zylindrischen Hohlraum.

J. Szabó

Institut für theoretische Physik der Universität - Budapest

(ricevuto il 15 Aprile 1960)

Ein zylindrischer Hohlraum von beliebigem Querschnitt und von der Länge l ist mit barotroper Flüssigkeit gefüllt, deren elektrische Leitfähigkeit unendlich groß, ihre Viskosität aber vernachlässigbar klein ist. Es werden die möglichen magnetohydrodynamischen Schwingungsformen im Fall untersucht, wo das homogene äußere magnetische Feld parallel zur Erzeugenden des Zylinders steht. Die z-Achse des Koordinatensystems steht in der Richtung des äußeren magnetischen Feldes (Abb. 1). Beschränken wir uns auf Schwingungen mit kleinen Amplituden, so ergibt sich leicht aus den Grundgleichungen der Magnetohydrodynamik [1]

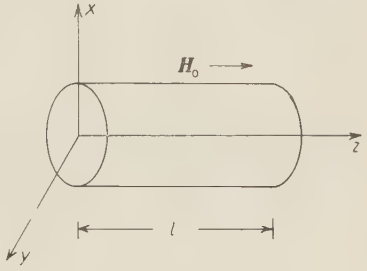


Abb. 1.

$$\frac{\partial^2 \boldsymbol{v}}{\partial t^2} - c_S^2 \operatorname{grad} \operatorname{div} \boldsymbol{v} - c_A^2 \boldsymbol{n} \times \operatorname{rot} \operatorname{rot} (\boldsymbol{n} \times \boldsymbol{v}) = 0 \,, \tag{1}$$

wo $\boldsymbol{v}$ die hydrodynamische Geschwindigkeit, $c_S = (\partial p / \partial \varrho)^{\frac{1}{2}}$ die Schallgeschwindigkeit, $c_A = H_0 (4\pi \varrho_0)^{-\frac{1}{2}}$ die Alfvénsche Geschwindigkeit, $\boldsymbol{n}$ aber der Einheitsvektor parallel zum magnetischen Feld ist. Im Fall ruhender Wand soll die Lösung $\boldsymbol{v}$ von (1) der folgenden Randbedingung genügen:

$$\boldsymbol{v} \cdot \boldsymbol{\nu}|_F = 0 \,, \tag{2}$$

wo $\boldsymbol{\nu}$ der Normalvektor der Wand ist, F aber die Randfläche des Hohlraums bedeutet.

---

[1] Siehe z.B., R. Gajewski: *The Physics of Fluids*, **2**, 633 (1959).

Nach der Bestimmung der Geschwindigkeit kann man die Perturbation $\boldsymbol{h}$ des magnetischen Feldes in der Flüssigkeit aus der Gleichung

$$\frac{\partial \boldsymbol{h}}{\partial t} = H_0 \operatorname{rot} (\boldsymbol{v} \times \boldsymbol{n}) , \tag{3}$$

die elektrische Feldstärke $\boldsymbol{E}$ aber aus der Gleichung

$$\boldsymbol{E} = \frac{H_0}{c} \boldsymbol{n} \times \boldsymbol{v} \tag{4}$$

berechnen [2]. Ist die elektrische Leitfähigkeit der Wand unendlich groß, so muß die elektrische Feldstärke der Randbedingung

$$\boldsymbol{\nu} \times \boldsymbol{E}|_F = 0 \tag{5}$$

genügen.

Im Fall zeitlich harmonischer Schwingungen können wir schreiben

$$\boldsymbol{v}(x, y, z, t) = \boldsymbol{V}(x, y, z) \sin (\omega t - \delta) . \tag{6}$$

Mit diesem Ansatz bekommt man aus (1)

$$\omega^2 \boldsymbol{V} + c_S^2 \operatorname{grad} \operatorname{div} \boldsymbol{V} + c_A^2 \boldsymbol{n} \times \operatorname{rot} \operatorname{rot} (\boldsymbol{n} \times \boldsymbol{V}) = 0 . \tag{7}$$

Unter Berücksichtigung der Bedingungen (2), (5) und der Gleichung (4) ergibt sich, daß die Geschwindigkeit in der Form $\sin (p\pi/l)z$ von z abhängt ($p = 1, 2, 3, \ldots$). Die spezielle Lösung von (7) ist also

$$\boldsymbol{V}(x, y, z) = \bar{\boldsymbol{v}}(x, y) \sin \frac{p\pi}{l} z . \tag{8}$$

Wir zerlegen die Geschwindigkeit in transversale und longitudinale Komponenten

$$\boldsymbol{V} = \boldsymbol{V}_T + \boldsymbol{V}_L , \tag{9}$$

so, daß

$$\boldsymbol{V}_T \cdot \boldsymbol{n} = 0 , \qquad \boldsymbol{V}_L \times \boldsymbol{n} = 0 \tag{10}$$

sind. Nach (8) kann man schreiben

$$\boldsymbol{V}_T = \bar{\boldsymbol{v}}_T(x, y) \sin \frac{p\pi}{l} z ; \qquad \boldsymbol{V}_L = \boldsymbol{n} \bar{v}_L(x, y) \sin \frac{p\pi}{l} z , \tag{11}$$

wo $\bar{v}_L = |\bar{\boldsymbol{v}}_L|$ ist.

---

[2] Vgl. z.B., S. I. SYROVATSKI: *Usp. Fiz. Nauk*, **62**, 247 (1957).

Setzt man (11) in die Gleichung (7) ein, so führt die transversale bzw. longitudinale Komponente von (7) zu den Gleichungen

$$\left(\omega^2 - c_A^2 \frac{p^2\pi^2}{l^2}\right) \bar{\boldsymbol{v}}_T + (c_S^2 + c_A^2) \operatorname{grad} \operatorname{div} \bar{\boldsymbol{v}}_T = 0 , \tag{12}$$

$$\operatorname{grad} \bar{v}_L = 0 , \tag{13}$$

$$\left(\omega^2 - c_S^2 \frac{p^2\pi^2}{l^2}\right) \bar{v}_L = 0 , \tag{14}$$

$$\operatorname{div} \bar{\boldsymbol{v}}_T = 0 . \tag{15}$$

Wir besprechen die Lösungen des Gleichungssystems (12)-(15):

a) Ist

$$\omega_p = \frac{p\pi}{l} c_S , \qquad (p = 1, 2, 3, \ldots), \tag{16}$$

so ergibt sich aus (12)

$$\operatorname{rot} \bar{\boldsymbol{v}}_T = 0 . \tag{17}$$

Die der Randbedingung genügende Lösung von (15) und (17) ist

$$\bar{\boldsymbol{v}}_T = 0 . \tag{18}$$

Aus (13) folgt weiter

$$\bar{v}_L = \text{Konst} . \tag{19}$$

Aus (3) und (4) ergibt sich, daß in diesem Fall

$$\boldsymbol{h} = 0 , \qquad \boldsymbol{E} = 0 \tag{20}$$

sind. Es ergeben sich also reine longitudinale akustische Schwingungen mit den Frequenzen

$$\nu_p = \frac{p}{2l} c_S . \qquad (p = 1, 2, 3, \ldots). \tag{21}$$

Die allgemeine Lösung von (1) ist also in diesem Fall

$$\boldsymbol{v}_L = \boldsymbol{n} \sum_p K_p \sin (p\pi/l) z \sin \left(\frac{p\pi}{l} c_S t - \delta_p\right) . \tag{22}$$

b) Ist nun

$$\omega_p = \frac{p\pi}{l} c_A , \tag{23}$$

so bekommt man aus (14)

$$\bar{\boldsymbol{v}}_L = 0 \,. \tag{24}$$

Aus (15) aber ergibt sich

$$\bar{\boldsymbol{v}}_T = \operatorname{rot} \boldsymbol{A}(x, y) = \operatorname{rot} [A(x, y)\boldsymbol{n}] = \operatorname{grad} A(x, y) \times \boldsymbol{n} \,. \tag{25}$$

Aus (2) folgt, daß $A(x, y)$ der Randbedingung

$$A(x, y)\,|_C = \text{Konst}\,, \tag{26}$$

genügen muß, wo C die Schnittkurve der Wand des Hohlraums mit der (x, y)-Ebene ist. Die Konstante in (26) können wir ohne Beschränkung der Allgemeinheit gleich Null wählen.

Die magnetische bzw. elektrische Feldstärke in der Flüssigkeit kann man leicht aus (3) bzw. (4) berechnen.

Es sind also reine transversale Schwingungen möglich mit den Frequenzen

$$\nu_p = \frac{p}{2l} c_A \,. \qquad (p = 1, 2, 3, \ldots). \tag{27}$$

Wir bemerken, daß man im Fall kreisförmigen Querschnittes mit Radius r_0 die Funktion A in (25) folgendermaßen darstellen kann:

$$A(r, \varphi) = \sum_{m,\nu} J_m\left(\frac{\lambda_{m\nu}}{r_0} r\right) [A_{m\nu} \cos m\varphi + B_{m\nu} \sin m\varphi]\,, \tag{28}$$

wo r und φ die Polarkoordinaten sind, $J_m(u)$ die Besselsche Funktion m-ter Ordnung, $\lambda_{m\nu}$ aber die ν-te Nullstelle von $J_m(u)$ bedeuten und $m = 0, 1, 2, \ldots$ ist.

Die den Randbedingungen genügende allgemeine Lösung von (1) läßt sich nach (25) und (28) in zylindrischen Koordinaten in der foldenden Form schreiben:

$$\begin{cases} \bar{v}_{Tr} = \displaystyle\sum_{m,\nu,p} \frac{1}{r} J_m\left(\frac{\lambda_{m\nu}}{r_0} r\right) m [B_{m\nu} \cos m\varphi - A_{m\nu} \sin m\varphi] \sin \frac{p\pi}{l} z \sin\left(\frac{p\pi}{l} c_A t - \delta_p\right), \\ \bar{v}_{T\varphi} = -\displaystyle\sum_{m,\nu,p} \frac{\lambda_{m\nu}}{r_0} J'_m\left(\frac{\lambda_{m\nu}}{r_0} r\right) [A_{m\nu} \cos m\varphi + B_{m\nu} \sin m\varphi] \sin \frac{p\pi}{l} z \sin\left(\frac{p\pi}{l} c_A t - \delta_p\right), \\ v_{Tz} = 0 \,, \end{cases} \tag{29}$$

wo $J'_m(u)$ die Ableitung der Funktion $J_m(u)$ nach u bedeutet.

Wenn der Hohlraum einen rechtwinkligen Querschnitt mit den Längen a und b hat, so ist

$$A(x, y) = \sum_{m_1, m_2} A_{m_1, m_2} \sin \frac{m_1 \pi}{a} x \sin \frac{m_2 \pi}{b} y \,. \qquad (m_1, m_2 = 1, 2, 3, \ldots): \tag{30}$$

Die rechtwinkligen Komponenten der allgemeinen Lösung $\boldsymbol{v}$ von (1) sind in diesem Fall nach (25) und (30)

$$(31)\quad \begin{cases} v_{Tx} = \sum\limits_{m_1,m_2\,p} A_{m_1,m_2} \dfrac{\pi}{b} m_2 \sin\dfrac{m_1\pi}{a} x \cos\dfrac{m_2\pi}{b} y \sin\dfrac{p\pi}{l} z \sin\left(\dfrac{p\pi}{l} c_A t - \delta_p\right), \\ v_{Ty} = -\sum\limits_{m_1,m_2,p} A_{m_1,m_2} \dfrac{\pi}{a} m_1 \cos\dfrac{m_1\pi}{a} x \sin\dfrac{m_2\pi}{b} y \sin\dfrac{p\pi}{l} z \sin\left(\dfrac{p\pi}{l} c_A t - \delta_p\right), \\ v_{Tz} = 0\,. \end{cases}$$

c) Im Fall, wo

$$\omega_p \neq \frac{p\pi}{l} c_S\,, \qquad \omega_p \neq \frac{p\pi}{l} c_A\,,$$

sind, kommt man auf die triviale Lösung von (12)-(15)

$$\bar{\boldsymbol{v}}_T = 0\,, \qquad \bar{\boldsymbol{v}}_L = 0\,.$$

Die erlaubten Schwingungen sind also entweder rein longitudinal mit den Frequenzen (21) (akustische Wellen), oder rein transversal mit den Frequenzen (27) (Alfvénsche Wellen), während in einem unendlich langen Wellenleiter die fortschreitenden magnetohydrodynamischen Wellen gleichzeitig beide Komponenten haben können.

* * *

Der Verfasser ist Herrn Cs. Hargitai für wertvolle Bemerkungen zu großem Dank verpflichtet.

Proton-Proton Differential Cross Section at 1 GeV.

J. D. Dowell, W. R. Frisken, G. Martelli (*),
B. Musgrave, H. B. van der Raay and R. Rubinstein
Physics Department, University of Birmingham - Birmingham

(ricevuto il 24 Giugno 1960)

Preliminary results are reported from the first of a new series of counter experiments which are being performed in this laboratory to investigate nucleon-nucleon interactions at 1 GeV. The p-p elastic differential cross-section in the angular range from 18° to 90° in the C.M. system has been measured at (1010 ± 20) MeV, using a polythene-carbon difference method. Measurements at this energy have already been reported by several authors [1]; the present experiment, by extending the angular range down to 18° C.M. (7° 30′ lab.), enables a more accurate estimate of the elastic cross-section to be made. The errors, based on counting statistics only, are less than 5% on all points.

This experiment has been made possible by the successful extraction of the beam of the Birmingham proton synchrotron [2], which provides 10^8 protons per pulse, focussed to a 1 in. diameter circle at the target position. The pulse duration is 4 ms.

The scattered and recoil protons were detected as triple coincidences between a defining telescope and a single large recoil counter, the resolving time being $6 \cdot 10^{-9}$ s. The angular resolution was about $\pm 1.5°$ in the C.M. system.

The results, corrected for accidental coincidences, are given in Table I. Corrections for inelastic p-p events and attenuation of the recoil protons are not included, but have been estimated to be less than 1% on all points.

The results have been fitted with curves of the form

$$\sigma(\theta) = \sum_{i=0}^{N} a_{2i} P_{2i}(\cos\theta) ,$$

(*) On leave of absence from the University of Pisa, Italy.

(1) L. W. Smith, A. W. McReynolds and G. Snow: *Phys. Rev.*, **97**, 1186 (1955); P. J. Duke, W. O. Lock, P. V. March, W. M. Gibson, J. G. McEwen, I. S. Hughes and H. Muirhead: *Phil. Mag.*, **2**, 204 (1955); A. P. Batson, B. B. Culwick, J. G. Hill and L. Riddiford: *Proc. Roy. Soc.*, A **251**, 218 (1959); J. D. Dowell, W. R. Frisken, G. Martelli and B. Musgrave: *Proc. Phys. Soc.*, **74**, 625 (1959).

(2) G. A. Doran, E. A. Finlay, H. R. Shaylor and M. M. Winn: *Nucl. Instr.*, to be published.

TABLE I.

c.m. angle	$\sigma(\theta)$ (mb/sr)	fitted curve
18° 30′	11.80 ±0.26	12.35
24° 36′	10.40 ±0.33	10.00
30° 42′	7.89 ±0.18	7.64
36° 42′	5.76 ±0.09	5.59
41° 30′	4.11 ±0.09	4.24
48° 30′	2.718±0.051	2.789
60° 0′	1.546±0.041	1.488
71° 6′	0.922±0.029	0.942
79° 48′	0.739±0.037	0.715
90° 0′	0.608±0.030	0.617

using a least squares method (*). The best fit has been obtained for $N=3$. The values of the coefficients a_{2i} and their errors are

$$a_0 = 3.314 \pm 0.053 ,$$

$$a_2 = 7.821 \pm 0.197 ,$$

$$a_4 = 4.113 \pm 0.240 ,$$

$$a_6 = 1.052 \pm 0.176 .$$

In Fig. 1 this best fit curve is shown together with the experimental points.

The absolute values of the differential cross-section have been obtained by normalizing the intercept of the fitted curve at 0° to a value of

$$\sigma(0°) = (16.30 \pm 0.35) \text{ mb/sr} .$$

This value has been derived from the relation

$$\text{Im } f(0°) = k\sigma_T/4\pi ,$$

using the result of (46.1±0.5) mb for σ_T, found in this laboratory by M. E. LAW, G. W. HUTCHINSON and D. H. WHITE (3).

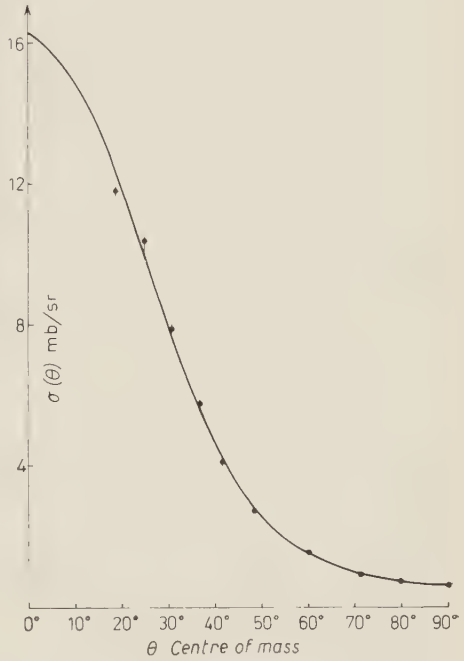


Fig. 1. – Least squares fit of the function

$$\sigma(\theta) = \sum_{i=0}^{3} a_{2i} P_{2i}(\cos\theta) ,$$

to the experimental points, normalized to $\sigma(0°)=(16.3\pm0.35)$ mb/sr derived from the relation $\text{Im } f(0°) = k\sigma_T/4\pi$.

This procedure assumes a negligible contribution from the real part of the forward scattering amplitude. The elastic cross-section, found by integration, is (20.8±1.0) mb. The error involved in the normalization has been taken into account in the value quoted for the elastic cross-section.

(*) The authors are deeply indebted to Dr. L. CASTILLEJO of the Department of Mathematical Physics of this University, for programming the least squares fit and for the many helpful discussions.

(3) M. E. LAW, G. W. HUTCHINSON and D. H. WHITE: *Nucl. Phys.*, **9**, 600 (1959).

IL NUOVO CIMENTO Vol. XVIII, N. 4 16 Novembre 1960

Note on Ending Particles in Nuclear Emulsions Exposed to the Primary Cosmic Radiation.

C. J. Waddington

H. H. Wills Physical Laboratory, University of Bristol - Bristol

(ricevuto il 26 Agosto 1960)

In a previous paper (1), it was shown that cosmic ray intensity variations might be studied by examining the rate of production of ending particles, ρ, in nuclear emulsions exposed to the primary cosmic radiation. The purpose of this note is to report on some further measurements of ϱ and to emphasize some of the precautions that are necessary if this technique is to yield the maximum amount of information.

Measurements of ε, the density of ending particles, have been made in eight additional emulsion stacks, while further measurements have also been made on some of those stacks referred to previously. Details are given in the Table I. The errors shown on ε are statistical, while those on ε_{corr}, the value of ε corrected for those particles produced during storage at sea level and during air transportation, include a $(20 \div 30)\%$ error on the magnitude of these corrections. It will be noticed that while these uncertainties in the corrections are relatively unimportant for stacks having high values of ε, *i.e.* stacks 1, 2 and 4, they are very serious when ε is low, *i.e.* stacks 15 and 17. It is most desirable, therefore, that workers intending to use this method should either have control emulsion with each stack, or, less satisfactory, that they should know and record the precise history of each stack from manufacture to development.

The values of ϱ obtained here are shown in the figure, plotted as a function of the cut-off rigidity, R, in GeV/c. Also shown are the relationships between ϱ and R deduced previously for « solar minimum » and « solar maximum », where the separation between the two periods was assumed to be at the beginning of 1957. The additional data shown in this figure suggest that, as might be expected, such a simple division into two separate classes is no longer justifiable. While the relationship derived for solar minimum still appears to be valid, it is apparent that since 1957 appreciable fluctuations have occurred such that no unique relation between ϱ and R exists for the whole of the period 1957-1959, even though such a relation must exist at any one time.

The table also gives the Chicago neutron monitor nucleon intensity re-

(1) C. J. Waddington: *Nuovo Cimento*, **14**, 1205 (1959).

Stack no. (*)	Place	Cut-off rigidity (GeV/c)	Date of exposure	ε (enders/cm^3)	% error	ε_{corr} (enders/cm^3)	% error	Flight (**) duration (h)	ϱ (enders/cm^3·h)	Nucleon intensity	
										bi-hourly values (N_p)	nat. log. (***)
1 (*O*)	Manitoba	0.6	3- 8-58	7 000	3.7	6 050	5.6	10.4	580 ± 33	2 673	83
2 (*M*)	Minnesota	1.0	18- 9-56	10 000	3.7	8 830	5.2	9.3	950 ± 50	—	—
3 (*N*)	Minnesota	1.0	14- 6-58	6 100	4.7	4 910	8.4	11.08	435 ± 37	2 715	99
4	Minnesota	1.0	31- 8-57	9 750	5.5	8 800	6.6	17.5	502 ± 33	2 588	51
5	Minnesota	1.0	30- 7-57	6 350	5.6	5 400	7.8	10.4	520 ± 40	2 839	143
6	N. England	1.65	29- 7-59	2 940	5.7	2 720	7.1	5.64	486 ± 35	2 591	52
7	N. Missouri	1.80	28- 9-56	5 400	5.3	4 700	7.7	7.0	670 ± 51	—	—
8 (*B*)	S. England	2.40	9- 7-54	3 260	5.7	3 200	5.9	5.65	567 ± 33	—	—
9	Central Missouri	2.30	3- 8-58	6 400	5.6	5 450	7.7	10.16	535 ± 41	2 673	83
10 (*D*)	S. England	2.40	20-11-58	2 820	6.9	2 690	7.4	6.23	432 ± 33	2 772	120
11 (*E*)	N. Italy	4.6	14- 9-54	2 950	5.5	2 700	6.3	6.75	400 ± 25	—	—
12 (*G*)	N. Italy	4.6	12-10-54	2 840	6.9	2 640	7.6	6.42	396 ± 31	—	—
13 (*T*)	Texas	4.5	8- 2-59	7 170	3.3	5 800	6.9	20.08	288 ± 20	2 719	100
13*a*	Texas	4.5	8- 2-59	7 450	5.2	6 500	6.9	20.08	323 ± 22	2 719	100
14	Texas	4.5	19-10-57	3 400	5.4	2 450	11.8	9.6	255 ± 30	2 792	127
15	Texas	4.5	26- 3-58	2 210	5.2	1 260	19.8	6.23	202 ± 40	2 513	21
16 (*S*)	Sardinia	5.6	29- 7-53	2 720	6.6	2 480	7.7	8.0	309 ± 25	—	—
17	Cuba	8.8	20- 3-58	2 570	5.8	1 570	17.0	7.17	220 ± 37	2 702	94

(*) Stacks considered previously have designation shown in brackets.
(**) Corrected for ascent and descent.
(***) $= \ln (N_p/1000) - 0.900$.

corded during the exposure of those stacks exposed after the commencement of the IGY. Examination of these data suggests that there is no detailed relation between ϱ and the nucleon intensity at sea level. Stacks 4 and 15 were both exposed during large Forbush decreases, but their ϱ values do not appear to be appreciably different from those obtained in comparable stacks flown at periods of « normal » intensity. A similar effect is apparent in stack 6, where, although the nucleon intensity was anomalously low, ϱ was high, and the flux of primary α-particles is also high (2). In view of the sensitivity of ϱ on the change in intensity from solar minimum to solar maximum this result appears difficult to explain if it is assumed that a Forbush decrease consists solely of a diminution in intensity of particles of all energies.

Finally it may be noted that the additional data presented here show that the relationships between ϱ and the primary α-particles flux must be modified from those given previously. The final form of these relations must await further comparable measurements of ϱ and α-particle fluxes.

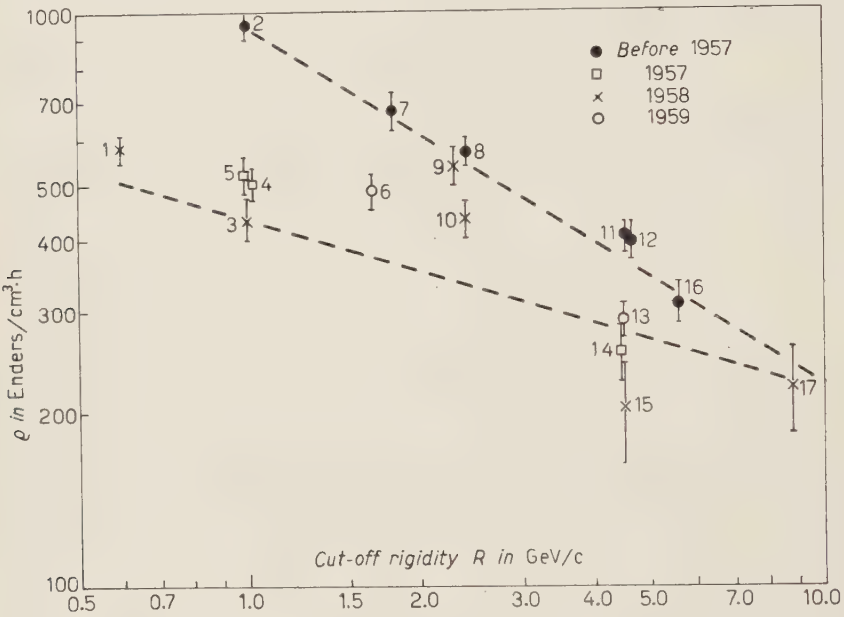


Fig. 1. – The variation of ϱ with R. The upper curve represents the relation derived previously for solar minimum, the lower that for solar maximum.

* * *

The experimental results reported here were predominantly obtained while visiting Washington University, St. Louis, and the author is deeply grateful to Dr. M. W. Friedlander for the hospitality and facilities of his laboratory, as well as for permission to make measurements on stacks 5, 9, 1 and 13*a*. He is also greatly indebted to Dr. P. S. Freier, of the University of Minnesota, for the loan of emulsions from stacks 4, 7, 14, 15 and 17.

(2) G. Stevenson and C. J. Waddington: unpublished (1960).

IL NUOVO CIMENTO Vol. XVIII, N. 4 16 Novembre 1960

Field-Theoretical Derivation of the Two-Body Potential (*).

S. N. Gupta

Department of Physics, Wayne State University - Detroit, Mich.

(ricevuto il 10 Ottobre 1960)

We have recently described a general method (1) of obtaining the two-body potential from the unitary expansion of the scattering operator. We have also applied this method to the derivation of the static two-pion exchange nuclear potential for non-relativistic velocities of the nucleons, taking fully into account the effect of the nucleon recoil. In a paper in this journal Charap and Tausner (2) have expressed doubts about the validity of dropping the velocity-dependent terms in the denominators of the contribution of the uncrossed two-pion exchange diagram. We shall, however, show that the neglect of all velocity-dependent terms in our derivation of the non-relativistic nuclear potential can be rigorously justified. We shall use the same notation as in our earlier paper, whose equations will be referred to as (I.1), (I.2), etc.

In order to derive the two-body potential up to the n-th order, we first have to find the scattering matrix elements $S_2, S_4, \ldots, S_n$, and then obtain $K_2, K_4, \ldots, K_n$ by the relations

$$K_2 = iS_2 , \tag{1}$$

$$K_4 = iS_4 - \tfrac{1}{2} iS_2 S_2 , \tag{2}$$

and so on. The relation (2) shows that K_4 is obtained by subtracting the iteration of the second-order matrix element iS_2 from the fourth-order matrix element iS_4. But, in practice it is not at all necessary to calculate the iteration of the second-order matrix element for the derivation of the fourth-order potential, because by taking the Hermitian conjugate of (1) and (2) and remembering that K_2 and K_4 are Hermitian, we get

$$K_2 = iS_2 = -iS_2^* , \tag{3}$$

$$K_4 = iS_4 - \tfrac{1}{2} iS_2 S_2 = -iS_4^* + \tfrac{1}{2} iS_2^* S_2^* . \tag{4}$$

(*) Supported in part by the U. S. National Science Foundation.

(1) S. N. Gupta: *Phys. Rev.*, **117**, 1146 (1960).

(2) J. M. Charap and M. J. Tausner: *Nuovo Cimento*, **18**, 316 (1960).

The above relations enable us to express K_4 as

$$K_4 = \tfrac{1}{2}i(S_4 - S_4^*)\,, \tag{5}$$

so that K_4 can be obtained simply by dropping the non-Hermitian terms in iS_4.

We shall further show that the Hermitian part of iS_4 gives us a purely static pion-theoretical nuclear potential for non-relativistic velocities of the nucleons. A velocity dependence of the two-pion exchange nuclear potential in the non-relativistic approximation can possibly arise only from the $(p \quad q)^2$ terms in the denominators of (I.15) and (I.16), where $\frac{1}{4}(p-q)^2 = \boldsymbol{p}^2$ represents the square of the propagation vector of each nucleon, and (I.15) and (I.16) represent the contributions of the uncrossed and the crossed two-pion exchange diagrams respectively. It is evident that the $(p-q)^2$ terms can be neglected in the denominators of (I.16) when $\boldsymbol{p}^2 \ll \varkappa^2$, but the neglect of similar terms in (I.15) is not self-evident. Therefore, we retain these terms in (I.15), and thus obtain for the contribution of the uncrossed diagram in the non-relativistic approximation after some simplification [3]

$$S_4(\alpha) = (i\pi^2 g^4/c^2\hbar^2)\,\delta(p - p' + q - q')\,[\psi^{*-}(\boldsymbol{q}')\tau_i\tau_j\psi^{+}(\boldsymbol{q})][\psi^{*-}(\boldsymbol{p}')\tau_i\tau_j\psi^{+}(\boldsymbol{p})]I_\alpha\,, \tag{6}$$

where

$$I_\alpha = \int_0^1 \mathrm{d}u \int_0^u \mathrm{d}v\, v \int_0^1 \mathrm{d}w \left(\frac{1}{2A} - \frac{\varkappa^2 v^2 w^2}{A^2}\right), \tag{7}$$

with

$$A = \varkappa^2 v^2 w^2 + \lambda^2(1-v) + \boldsymbol{k}^2(u-v)(1-u) - \boldsymbol{p}^2 v^2\,. \tag{8}$$

Since

$$\frac{\partial}{\partial w}\left(\frac{w}{2A}\right) = \frac{1}{2A} - \frac{\varkappa^2 v^2 w^2}{A^2}\,,$$

we can immediately carry out the integration over w in (8), and obtain

$$I_\alpha = \int_0^1 \mathrm{d}u \int_0^u \mathrm{d}v\, v\, \frac{1}{\left[\varkappa^2 v^2 + \lambda^2(1-v) + \boldsymbol{k}^2(u-v)(1-u) - \boldsymbol{p}^2 v^2\right]^{\frac{1}{2}}}\,. \tag{9}$$

For $\boldsymbol{p}^2 \ll \varkappa^2$, (9) gives us

$$I_\alpha = \int_0^1 \mathrm{d}u \int_0^u \mathrm{d}v\, v\, \frac{1}{\left[\varkappa^2 v^2 + \lambda^2(1-v) + \boldsymbol{k}^2(u-v)(1-u)\right]^{\frac{1}{2}}}\,, \tag{10}$$

which leads to a purely static non-relativistic nuclear potential in agreement with the result of our earlier paper.

---

[3] It should be observed that the leading terms in $S_4(\alpha)$ correspond to the values $\mu = \nu = 4$ of the indices μ and ν in (I.15). It is also interesting to note that in the non-relativistic approximation $S_4(\alpha)$ becomes purely Hermitian, because the non-Hermitian part of $S_4(\alpha)$ consists of lower order terms.

IL NUOVO CIMENTO Vol. XVIII, N. 4 16 Novembre 1960

The Scattering Constant for Multiply Charged Particles in Photographic Emulsion.

C. Fichtel and M. W. Friedlander

(*Nuovo Cimento*, **10**, 1032 (1958))

ERRATA

Figs. 1 and 2 illustrate the variation of scattering constant with cell-size and velocity for protons and α-particles.

The labelling of the curves with respect to velocity has been inverted: the curves labelled $\beta=1.0$ should be labelled $\beta=0.5$, and vice versa for the curves labelled $\beta=0.5$.

LIBRI RICEVUTI E RECENSIONI

K. M. Koch und W. Jellinhaus – *Einführung in die Physik der Magnetischen Werkstoffe*. Deutiche, Wien, 1957; pp. VII+208. Prezzo: Scellini Austriaci 186 (= DM. 31).

Il volume fa parte di una Collana sulla Fisica dei vari materiali che interessano la Elettrotecnica, che l'Editore Deutiche ha cominciato a pubblicare da qualche anno. La materia vi è divisa in 7 capitoli, secondo uno schema ben organizzato.

Il I cap. è dedicato alle questioni dei elettrologia generale che più direttamente interessano il campo magnetico e la magnetizzazione della materia. Nel II cap. sono esposte le basi delle teorie atomiche con le quali si interpretano i vari stati magnetici della materia, soffermandosi in particolare sulle trattazioni che il ferromagnetismo ha avuto, da Weiss fino alle moderne teorie quantistiche.

Nel III e IV cap. viene descritto il comportamento dei ferromagnetici nei fondamentali fenomeni che oggi appaiono caratteristici di esso. Partendo dalle proprietà dei monocristalli, e tenendo conto della struttura a domini, vengono successivamente esaminati la magnetostrizione, l'influenza delle tensioni elastiche, la magnetoresistenza, per passare poi ad una interpretazione fenomenologica di tutte le proprietà della curva di magnetizzazione dei ferromagnetici tecnici. Un altro capitolo è dedicato al comportamento dei materiali in campo magnetico debole o forte, e a quello nei campi ad alta frequenza.

Infine gli ultimi due capitoli, coll'aiuto di alcune tabelle, forniscono una sintetica rassegna delle fondamentali caratteristiche tecnologiche che differenziano i principali materiali impiegati nella elettrotecnica. Vi si trova pure un'idea, sia pure generica, dei metodi di misura del campo e della induzione, sia nel caso di magnetizzazione costante, come in quello di magnetizzazione alternata.

Ottima la presentazione tipografica.

A. Drigo

K. M. Koch und R. Reinbach – *Einführung in die Physik der Leiterwerkstoffe*. Deutiche, Wien, 1960; pp. VII+255. Prezzo: Scellini Austriaci 222.

Questo volume, che fa pure parte della Serie sulla Fisica dei materiali impiegati nell'elettrotecnica, costituisce una interessante sintesi delle conoscenze sulla fisica della conduzione elettrica oggi indispensabili anche agli elettrotecnici per un razionale sfruttamento dei vari materiali che la tecnica moderna mette a loro disposizione.

Dopo una breve introduzione sulla conduzione ionica o elettronica che può

caratterizzare i vari tipi di solidi, dato un cenno ai modelli costitutivi di questi vengono giustificate in base alla classica teoria elettronica dei metalli le principali caratteristiche della loro conduzione elettrica, dalla legge di Ohm, alla dipendenza termica della conducibilità, fino alla sopraconducibilità. Un cenno è pure dato al problema dei semiconduttori, e al modello delle bande di energia nei solidi.

Coi medesimi modelli classici sono pure interpretati l'effetto termoelettrico e i fenomeni Peltier e Thomson, e un apposito capitolo è riservato agli effetti del campo magnetico sulla conduzione, al fenomeno di Hall, e ai vari effetti galvano- e termo-magnetici.

Un altro capitolo è dedicato infine alle basi delle moderne teorie sul meccanismo della conduzione, dalle trattazioni statistiche di Boltzmann e Fermi a quelle fondate sulla meccanica ondulatoria. Con queste ultime vengono rappresentate anche le interazioni fra elettroni e reticolo.

Utile infine il complesso di dati fisici e tecnologici sui materiali conduttori di maggiore interesse per l'elettrotecnica, che sono suddivisi in gruppi, da quello dei conduttori propriamente detti, a quello delle leghe per resistenze e per riscaldamento. Particolareggiata è pure la trattazione concernente i materiali per termometri a resistenza, e quelli per termocoppie. Buona, come sempre, la veste tipografica.

A. Drigo

J. Mikusinski – *Operational calculus*. International Series of Monographs on Pure and Applied Mathematics, vol. 8. Pergamon Press Ltd., London, 1959, pp. 495, prezzo 5 sterline.

Quest'opera di J. Mikusinski, già apparsa in polacco (2 edizioni), in russo e in tedesco, viene ora pubblicata in lingua inglese. L'attuale edizione differisce dalle precedenti perchè contiene in più un'appendice di oltre 100 pagine che costituisce la parte VI del volume.

Si tratta di un libro interessante soprattutto per l'originale impostazione data al calcolo degli operatori. L'esposizione è mantenuta su un piano molto elementare (salvo che nella predetta appendice) e la teoria è illustrata da numerose applicazioni concrete, cosicchè il volume può interessare una vasta cerchia di lettori.

L'opera è divisa in 7 parti.

La parte I tratta dell'algebra degli operatori: si considera l'anello delle funzioni $a(t)$ continue per $t \geqslant 0$ nel quale la somma è quella ordinaria, mentre il prodotto è il *Faltung* definito da

$$a(t)\,b(t) = \int_0^t a(\tau)\,b(rt - \tau)\,\mathrm{d}\tau;$$

si estende tale anello ad un campo e gli elementi di questo sono chiamati operatori. Veramente non viene usato questo linguaggio algebrico, ma si presenta la definizione di operatore in modo elementare paragonando le funzioni ai numeri interi e gli operatori ai numeri razionali. Con pochi semplici sviluppi di quest'idea (e considerando anche funzioni continue a tratti, sommabili in ogni intervallo limitato) si arriva subito ad interessanti applicazioni alle reti di circuiti elettrici e ad alcuni problemi sulle travi.

Nella parte II si sviluppa la teoria delle successioni e delle serie di operatori con applicazioni varie (per esempio alcuni tipi di equazioni alle differenze).

La parte III è dedicata al calcolo differenziale degli operatori, con molte applicazioni (corde vibranti, propagazione del calore, equazione dei telegrafisti).

Fino a questo punto le applicazioni alle equazioni differenziali, ordinarie o a derivate parziali, son fatte soltanto su esempi particolari; una trattazione più generale si trova nella parte IV.

La parte V è rivolta al calcolo integrale degli operatori: sono mostrate varie applicazioni e, fra l'altro, si fa un confronto dei metodi usati nel libro con quelli usati fondati sull'uso della trasformazioni di Laplace.

La parte VI è l'appendice di cui si è già detto. In essa sono esposti vari complementi alle parti I, II, IV, V: dimostrazioni di teoremi omesse in tali parti; esposizione di risultati recenti sul calcolo operazionale; studio delle relazioni fra tale calcolo e l'algebra astratta la teoria delle distribuzioni e gli spazi di Banach; indicazione di problemi non ancora risolti.

Infine la parte VII è una raccolta di formule e tabelle, destinate soprattutto alle applicazioni elettrotecniche.

Si può dire nel complesso che questo volume differisce dai numerosi trattati sul calcolo simbolico già esistenti, soltanto nell'impostazione, giacchè la parte formale risulta essere la medesima; comunque essa costituisce un altro (ben riuscito) esempio dei moderni tentativi di algebrizzare ogni teoria. Riteniamo però che il metodo solito della trasformata di Laplace sia concettualmente più semplice e perspicuo.

A. Ghizzetti

Direttore responsabile: G. POLVANI Tipografia Compositori - Bologna

Questo fascicolo è stato licenziato dai torchi il 22-XI-1960